1990.

1/ Assume $t = k d^\alpha \rho^\beta g^\gamma \mu^\delta$ where k is dimensionless

then $T = L^\alpha M^\beta L^{-3\beta} L^\gamma T^{-2\gamma} M^\delta L^{-\delta} T^{-\delta}$

and $t = k d^{1/2 - 3\delta/2} \rho^{-\delta} g^{-1/2 - \delta/2} \mu^\delta$

$= \sqrt{\frac{d}{g}}\, k \left(\frac{\mu}{\rho d^{3/2} g^{1/2}}\right)^\delta$

so one expression for G is $\frac{\mu}{\rho d^{3/2} g^{1/2}}$

If liquid is inviscid $\delta = 0$ so $t = k\sqrt{\frac{d}{g}}$

2/ i) $x = 0$

ii) put $y = x^\lambda$ indicial equation $(3\lambda+1)^2 = 0$

general solution $y = (A + B\log_e x)x^{-1/3}$

iii) let $z = -x$ eqn $3z^2\frac{d^2y}{dz^2} + 5z\frac{dy}{dz} + \frac{1}{3}y = 0$.

soln $y = z^{-1/3}(A + B\log_e z)$

$= (-x)^{-1/3}(A + B\log_e z)$

3/ $y = \sum_{j=0}^\infty a_j x^j$

so $\sum_{j=0}^\infty (j+2)(j+1)a_{j+2}x^j + \sum_{j=0}^\infty (j+1)a_{j+1}x^j + \sum_{j=0}^\infty a_j x^j + \sum_{j=1}^\infty a_{j-1}x^j = \sum_{j=0}^\infty \frac{x^j}{j!}$

$= (2a_2 + a_1 + a_0 - 1) + (6a_3 + 2a_2 + a_1 + a_0 - 1)x + \sum_{j=2}^\infty\left((j+2)(j+1)a_{j+2} + (j+1)a_{j+1} + a_j + a_{j-1} - \frac{1}{j!}\right)x^j = 0$

$a_0 = 1$, $a_1 = 0$, $a_2 = 0$, $a_3 = 0$

$(j+2)(j+1)a_{j+2} = \frac{1}{k!} - a_{j-1} - a_j - (j+1)a_{j+1}$

$a_4 = \frac{1}{24}$ $a_5 = 0$

4/ i) $\rho\frac{d\underline{u}}{dt} = -\nabla p + \rho\underline{F}$

$\underline{u} = \underline{0}$ $\underline{F} = F\underline{e}_r$

so $\frac{\partial p}{\partial r} = \rho F$ $\frac{\partial p}{\partial \theta} = 0$ $\frac{\partial p}{\partial \phi} = 0$

last 2 give $p = p(r)$ so $\frac{dp}{dr} = \rho F$

ii) $\frac{dp}{dr} = -\frac{k\rho}{r^n}$ so $p = \frac{\rho k}{(n-1)r^{n-1}} + C$

$p \to 0$ as $r \to \infty$ so $C = 0$

At $r = a$ $p = \frac{k\rho}{a} = \frac{k\rho}{(n-1)a^{n-1}}$ so $n = 2$.

5/ i) strength $= UR^2\underline{k}$ at origin

ii) $\operatorname{div}\underline{u} = 0$

iii) $\nabla\wedge\underline{u} = \underline{0}$ $(r > 0)$ fluid inviscid, $\boxed{\text{incompressible}}$ conservative body forces so yes by Persistence of irrotational motion

iv) zero

6/ $c = \frac{1}{3}$ Using d'Alembert's solution

F given point $x - ct = 1$, $x + ct = 3$. Interval of dependence is on positive x axis so no extension needed $p(x) = \frac{e^{-x^2}}{12}$ $q(x) = \cos\pi x$

$u(2,3) = \frac{1}{24}(e^{-1} + e^{-9}) + \frac{3}{2}\int_1^3 \cos\pi s\, ds$

$= 0.0153$ (to 3sf)

7/ i) Put $v = \frac{\partial w}{\partial y}$ to give $\frac{\partial v}{\partial y} - \frac{1}{y}v = 0$

I F $= \frac{1}{y}$, $\frac{\partial w}{\partial y} = v = y f(x)$

$w = \frac{1}{2}y^2 f(x) + g(x)$ or $y^2h(x) + g(x)$.

ii) a) $B^2 - 4AC = 0$ $\therefore$ parabolic

b) $S = -\frac{x}{2y}$

c) $\frac{\partial u}{\partial x} = x\frac{\partial u}{\partial \xi}$ $\frac{\partial u}{\partial y} = 2y\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \phi}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \xi} + x^2\frac{\partial^2 u}{\partial \xi^2}$

$\frac{\partial^2 u}{\partial y^2} = 2\frac{\partial u}{\partial \xi} + 4y^2\frac{\partial^2 u}{\partial \xi^2} + 4y\frac{\partial^2 u}{\partial\phi\partial\xi} + \frac{\partial^2 u}{\partial \phi^2}$

$\frac{\partial^2 u}{\partial x\partial y} = 2yx\frac{\partial^2 u}{\partial \xi^2} + x\frac{\partial^2 u}{\partial\xi\partial\phi}$

equation becomes $\frac{\partial^2 u}{\partial \phi^2} - \frac{1}{\phi}\frac{\partial u}{\partial \phi} = 0$.

d) using (i) $u = g\left(\frac{x^2}{2} + y^2\right) + y^2 h\left(\frac{x^2}{2} + y^2\right)$

8. i) $F_r = \frac{u}{\sqrt{gD}}$

$[F_r] = \frac{LT^{-1}}{(LT^{-2}L)^{1/2}} = 1$. F_r is dimensionless

$F = \frac{\rho_a}{\rho_o - \rho_a}F_r^2$, Yes as $\left[\frac{\rho_a}{\rho_o - \rho_a}\right] = 1$

ii) a) Apply Bernoulli's equation (check conditions) to a streamline on surface of geyser. $P = P_0$ on surface

$\frac{1}{2}u^2 + gz = \text{const}$

when $z = 0$ $u = u_G$

when $z = h$ $u = u_h$

$\frac{1}{2}u_G^2 = \frac{1}{2}u_h^2 + gh$. (1)

b) Incompressible so by continuity

$\frac{\pi D_h^2 u_h}{4} = \frac{\pi D_G^2 u_G}{4}$

Sub in (1) $h = \frac{u_G^2}{2g}\left(1 - \frac{D_G^4}{D_h^4}\right)$

c) If $D_G << D_h$ $h = \frac{u_G^2}{2g}$ using i) gives answer

d) use expression for F to get

$u_G = \sqrt{\frac{g}{D_G}\frac{h}{1.85}}\left(1 - \frac{\rho_a}{\rho_o}\right)^{1/2}$

$= 78.2\ \text{ms}^{-1}$