PART II

Answer THREE *questions in this part.*

Each question carries 20% of the total examination marks.

Question 7

The motion of a vibrating string is modelled by the following equation:

$$64\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad (0 < x < 1, \quad t > 0),$$

subject to the conditions:

$$\left.\begin{aligned} u(0,t) &= 0 \\ \frac{\partial u}{\partial x}(1,t) &= 0 \end{aligned}\right\} \qquad (t > 0)$$

$$\left.\begin{aligned} u(x,0) &= \frac{x^2}{10} \\ \frac{\partial u}{\partial t}(x,0) &= \sin(2\pi x) \end{aligned}\right\} \qquad (0 < x < 1)$$

where $u(x,t)$ is the displacement of the position x at time t.

(i) Which end of the string is fixed and which end is free? What is the speed, c, of wave propagation? What function describes the string's initial displacement? [2]

(ii) Use d'Alembert's solution of the wave equation to find the displacement at the centre of the string, P, at time $t = 1$. [$7\frac{1}{2}$]

(iii) What is the latest time at which the displacement of P can be calculated without extending the domains of the functions representing the initial conditions of the string? [2]

(iv) The solutions $u(x,t)$ to the above problem are of the form $u(x,t) = X(x)T(t)$ where X is a solution of a boundary value problem with eigenvalues $\lambda_n = (n+\frac{1}{2})^2\pi^2$ and corresponding eigenfunctions $\sin(n+\frac{1}{2})\pi x$, $n = 0, 1, 2, \ldots$, and T must satisfy $T'' + \frac{\lambda}{64}T = 0$ with $\lambda = \lambda_n$.

Show that the displacement of the string at any position x and time t after it has been set in motion is given by

$$u(x,t) = \sum_{n=0}^{\infty}\left[c_n \sin\left\{\frac{(2n+1)\pi t}{16}\right\} + d_n \cos\left\{\frac{(2n+1)\pi t}{16}\right\}\right]\sin(n+\tfrac{1}{2})\pi x.$$

Show that the coefficients c_n and d_n are given by

$$c_n = \frac{32}{(2n+1)\pi}\int_0^1 \sin 2\pi x \sin(n+\tfrac{1}{2})\pi x \, dx$$

and

$$d_n = \frac{1}{5}\int_0^1 x^2 \sin(n+\tfrac{1}{2})\pi x \, dx,$$

but *do not evaluate* these integrals. [$8\frac{1}{2}$]

[You may assume that any function f for which f and its derivatives are piecewise-continuous on $0 < x < \ell$ can be represented by the series

$$\sum_{n=0}^{\infty} b_n \sin\frac{(n+\frac{1}{2})\pi x}{\ell}, \text{ where } b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{(n+\frac{1}{2})\pi x}{\ell}\,dx,$$

and that any term-by-term differentiation required is permissible.]