Question 5

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An incompressible inviscid fluid flows down a channel with speed U. Across the floor of the channel there is a small semi-cylindrical ridge of radius a where a is negligible compared with the depth of the water. Assuming that the flow can be modelled as being two-dimensional, the stream function is given by

$$\psi(r,\theta) = U(r - \frac{a^2}{r})\sin\theta$$

where r and θ are plane polar coordinates with the origin at the centre of the cylinder. Write down the velocity components of the flow. Assuming that the effect of gravity at the ridge can be considered constant show that the downward force per unit length on the cylinder is $2ap_o - \frac{5}{3}\rho U^2 a$, where p_o is the pressure at a long distance from the ridge.

(If needed, you may quote the results: $\int_0^\pi \sin^3\theta d\theta = \frac{4}{3}$ and $\int_0^\pi \sin\theta d\theta = 2$.) [7]

Question 6

As a result of two waves travelling in opposite directions along the length of a narrow lake, a standing wave oscillation known as a seiche is formed. The height of the water surface with respect to some equilibrium position, ζ, may be represented by the function

$$\zeta(x,t) = 0.3\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi ct}{L}\right) \qquad (0 \le x \le L,\ t \ge 0)$$

where L is the length of the lake (in metres).

(i) Show that $\zeta(x,t)$ satisfies the wave equation.

(ii) Are there values of x for which this mathematical model predicts no movement of the surface? If there are, give these values.

(iii) What is the maximum vertical displacement (the amplitude) at the shore ($x = 0$)? [ζ is measured in metres above its equilibrium position.]

(iv) The length of the lake, L, is 1500 metres and it is 10 metres deep. Use an appropriate approximation to find the wave speed, c, when the wavelength is twice the length of the lake and show that the period of oscillation of the wave is approximately 5 minutes. [You may take the magnitude of the acceleration due to gravity, g, as 10 m s^{-2}.]

[8]

Question 7

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(i) Find the general solution $w = w(x,y)$ of the partial differential equation

$$\frac{\partial^2 w}{\partial y^2} - \frac{1}{y}\frac{\partial w}{\partial y} = 0 \qquad (y \ne 0).$$

[3]

(ii) The function $u(x,y)$ satisfies the partial differential equation

$$4y^2\frac{\partial^2 u}{\partial x^2} - 4xy\frac{\partial^2 u}{\partial x\partial y} + x^2\frac{\partial^2 u}{\partial y^2} - \frac{4y^2}{x}\frac{\partial u}{\partial x} - \frac{x^2}{y}\frac{\partial u}{\partial y} = 0 \quad (x \ne 0, y \ne 0). \quad (1)$$

(a) Show that this equation is parabolic in the region R of the (x,y)-plane over which it is defined.

(b) Find the equations of the characteristic curves in the region R and hence show that the characteristic coordinates may be chosen to be

$$\zeta = \frac{x^2}{2} + y^2, \qquad \phi = y.$$

(c) Use these characteristic coordinates and the chain rule to transform the partial differential equation (1) to its standard form.

(d) Hence, and using the result of Part (i), above, find the general solution $u = u(x,y)$ of Equation (1).

[17]