Part I

Answer **ALL SIX** *questions in this part.*

The questions in this part are not all worth the same number of marks.

The number of marks assigned to each question is given in square brackets.

Part I as a whole carries 40% of the total examination marks.

Question 1

Use the change of variable $x = \sin t$ $(0 < t < \pi/2)$ to find the general solution, in terms of x, of the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0 \qquad (0 < x < 1).$$ [7]

Question 2

Find a set of characteristic curves for each of the following differential equations, in which u is a function of x and y:

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} - u = 0 \qquad (x \neq 0);$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} - y^2\dfrac{\partial^2 u}{\partial y^2} - u = 0 \qquad (x \neq 0,\ y \neq 0).$ [6]

Question 3

(i) A fluid of density ρ and coefficient of viscosity μ flows along a pipe of diameter d with a mean speed V_m. A constant pressure difference Δp is applied along a length L of the pipe. Use the method of dimensional analysis to show that the pressure gradient is given by

$$\frac{\Delta p}{L} = \frac{\rho V_m^2}{d} f(Re),$$

where f is an undetermined function of the Reynolds number $Re = \rho d V_m/\mu$.

(ii) Rearrange the Hagen–Poiseuille formula to find the specific form for f when the pressure gradient is constant. [7]

Question 4

A fluid flows radially in all directions from a point with a spherically symmetric velocity

$$\mathbf{u} = u_r(r,t)\mathbf{e}_r,$$

where r and $\mathbf{e}_r$ are respectively the radial distance and unit radial vector from the point and t is time.

(i) *Show* that the continuity equation for the fluid, except at the point itself, can be expressed in the form

$$\frac{\partial \rho}{\partial t} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + u_r\frac{\partial \rho}{\partial r} = 0 \qquad (r > 0),$$

where ρ is the density of the fluid.

(ii) Suppose now that the fluid is incompressible and the flow is steady. Find an expression for u_r in terms of r and show that the path of a particle that lies on the sphere $r = a$ and has speed $u_r = U$, at $t = 0$, is given by

$$r^3 = 3Ua^2t + a^3.$$ [7]

Question 5

A rectangular channel of width 3.0 m consists of two horizontal sections joined by a gradual slope so that the bottom of the downstream section is 0.06 m higher than that of the upstream section. The flow of water in the channel is steady and is at a depth of 0.09 m in the upstream section. The water surface rises over the raised section so that the depth of water in the downstream section is 0.12 m.

Assuming that water is an inviscid, incompressible fluid, determine the volume flow rate along the channel.

[Take the magnitude of the acceleration due to gravity as $10\,\mathrm{m\,s^{-2}}$.] [6]

Question 6

(i) The velocity vector field at time $t = 0$ for a fluid is

$$\mathbf{v} = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}.$$

A closed path C is formed from the straight lines joining $O(0,0)$, $A(1,0)$ and $B(1,1)$. These lines are given as follows:

$OA : y = 0$ and $x = \alpha \quad (0 \le \alpha \le 1)$

$AB : x = 1$ and $y = \beta \quad (0 \le \beta \le 1)$

and $\quad OB : x = y = \gamma \quad (0 \le \gamma \le 1)$.

By *evaluating* the line integral, show that the circulation $\oint_C \mathbf{v} \cdot d\mathbf{r}$ is equal to zero.

(ii) Show further that the circulation is zero when C is *any* closed path in the fluid at time $t = 0$. [7]

PART II

Answer* THREE *questions in this part.

Each question carries 20% of the total examination marks.

Question 7

(i) Consider the time-dependent flow of a fluid whose velocity vector field $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ with Cartesian components is

$$u_1 = -2y \quad \text{and} \quad u_2 = 2xt.$$

Is the flow (a) incompressible, (b) irrotational, (c) steady? Give reasons for your three answers. [5]

(ii) Write down the equations describing the stream function for the velocity vector field in Part (i). Hence find the stream function for this flow. Sketch some of the streamlines at $t = 1$, showing the direction of flow. [7]

(iii) Consider the flow of an inviscid fluid of constant density ρ given by the velocity vector field of Part (i) with body force (per unit mass)

$$\mathbf{F} = 2x\mathbf{j}.$$

Find the pressure distribution in the fluid (to within an arbitrary function of time) and hence show that the pressure along any streamline at $t = 1$ is constant. [8]

Question 8

A viscosity pump is made up of a stationary, cylindrical casing and a drum of diameter D that rotates inside the casing at a constant angular speed ω; the drum and the casing are concentric. Liquid with density ρ and coefficient of viscosity μ enters a section AA′, flows around the annulus between the drum and casing, and leaves at another section BB′ (see Figure 1a). The pressure at BB′ is higher than at AA′, by Δp. The gap of the annulus h is so small compared with the diameter of the drum that the flow can be considered as that between two flat, parallel plates of length L, as shown in Figure 1b.

Figure 1a

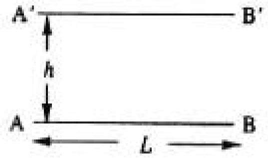

Figure 1b

Choose a Cartesian coordinate system so that the flow is in the x-direction and the lower plate AB lies in the plane $z = 0$.

(i) Assume that the flow is laminar and two-dimensional. What can you say about the y-component of the velocity vector $\mathbf{u} = (u_1, u_2, u_3)$, and the y-dependence of the other velocity components, u_1 and u_3?

Write down the boundary conditions that the velocity vector $\mathbf{u}$ must satisfy on the plates AB and A′B′. [3]

(ii) Assume that the flow is fully developed and that there are no body forces. Taken together with these assumptions, what else in the question statement suggests that the flow is steady? Write down the mathematical implications of steady flow.

Explain why it is reasonable to take $u_3 = 0$ everywhere.

Why is it reasonable to assume that the fluid is incompressible? Write down a mathematical statement of incompressibility involving $\mathbf{u}$. Hence show that $u_1 = u_1(z)$. [5]

(iii) Write down the x- and z- components of the Navier–Stokes equations for this flow. Solve them together with the boundary conditions of Part (i) to find the pressure p and velocity component u_1, and hence show that the flow rate (per unit depth in the y-direction) Q is given by

$$Q = \frac{\omega D h}{4} - \frac{\Delta p h^3}{12 L \mu}.$$

Deduce that Q is less than $\omega D h/4$. [12]

Question 9

(i) Show that the eigenvalue problem

$$X''(x) + \lambda X(x) = 0 \quad (0 < x < \pi),$$
$$X'(0) = X(\pi) = 0,$$

has eigenvalues $\lambda_n = (2n-1)^2/4$ and corresponding eigenfunctions

$$X_n(x) = \cos\left(\frac{2n-1}{2}\right) x \quad (n = 1, 2, \ldots).$$ [5]

(ii) A porous rod of length π containing moisture has one end sealed and the other end in contact with a dry medium, and loses moisture through its surface to dry air. The concentration of moisture $u(x,t)$ satisfies the problem given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} + \gamma^2 u \quad (0 < x < \pi,\ t > 0),$$

with

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad u(\pi,t) = 0 \quad (t \geq 0)$$

and

$$u(x,0) = \cos 2x \cos\frac{x}{2} \quad (0 < x < \pi),$$

where x and t represent distance along the rod and time, respectively, and k (> 0) and γ are constants.

(a) Use a trigonometric identity to show that the given initial condition can be expressed as a sum of cosine terms.

(b) Use the method of separation of variables and the result of Part (i) to determine the concentration of moisture in the rod at times subsequent to $t = 0$. [15]

Question 10

Plane sound waves propagate in a musical instrument (a pipe) of length ℓ. This situation can be modelled by the equation

$$\frac{\partial}{\partial x}\left[\frac{1}{A(x)}\frac{\partial}{\partial x}\left(A(x)u(x,t)\right)\right] - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t) = 0 \quad (0 < x < \ell, A(x) \neq 0), \qquad (1)$$

where $u(x,t)$ is the position at time t of an air particle whose equilibrium position is x, measured from one end of the instrument, $A(x)$ is the variable cross-sectional area of the pipe, and c is the speed of sound. The two ends of the instrument are open to the air so that there are free-end boundary conditions on u at these two ends.

(i) (a) Use the method of separation of variables to show, clearly, that there are solutions of Equation (1) of the form

$$u(x,t) = X(x)T(t)$$

for which $X(x)$ satisfies the Sturm–Liouville problem

$$\left.\begin{array}{l} X''(x) + f(x)X'(x) + (f'(x) + \lambda)X(x) = 0, \\ X'(0) = X'(\ell) = 0. \end{array}\right\} \qquad (2)$$

Give the expression for $f(x)$ in terms of $A(x)$.

(b) Find the eigenvalues and eigenfunctions of the problem given by Equations (2) in the case where $A(x)$ is a non-zero constant. [12]

(ii) Apply d'Alembert's solution to the problem of sound waves travelling in an infinitely long pipe of constant cross-section, that is to solving Equation (1) in the case where $A(x)$ is constant, when u is subject to the initial conditions:

$$\left.\begin{array}{ll} u(x,0) = \cos x & (-\infty < x < \infty), \\ \dfrac{\partial u}{\partial t}(x,0) = \sin x & (-\infty < x < \infty). \end{array}\right\} \qquad (3)$$

Show that the solution to this problem can be written as a sum of standing waves and confirm that it satisfies the initial conditions (3). Explain, *briefly*, how you would tackle the problem of finding $u(x,t)$ for waves travelling in a pipe (musical instrument) of finite length using a method involving d'Alembert's solution. [8]

Question 11

(i) Explain *briefly* what you understand by the following:

(a) *a fluid, shear stress* and *compressibility*, using your explanations to distinguish between solids, liquids and gases;

(b) *the Magnus effect*, giving at least two examples of where this may be observed or used in everyday life;

(c) *d'Alembert's paradox* and how it is resolved;

(d) *the incompressibility of a fluid and of a flow*, including a brief explanation of the nature of the volume flow rate of a liquid across a surface S enclosing a volume V. [15]

(ii) Sketch a graph of the variation of c^2 with h (≥ 0) where c is the speed of waves travelling in water of depth h. Use your graph to explain the refraction of water waves as they approach a sloping beach. [5]

[END OF QUESTION PAPER]