Part I

Answer **ALL SIX** *questions in this part.*

The questions in this part are not all worth the same number of marks.

The number of marks assigned to each question is given in square brackets.

Part I as a whole carries 40% of the total examination marks.

Question 1

(i) Starting from the Navier-Stokes equation for a fluid in motion under the influence of a body force per unit mass $\mathbf{F}$, derive the form of this equation for a fluid at rest.

(ii) Determine the pressure distribution due to gravity in a static solution of brine that has a variable density given by $\rho = a + bz$, where a and b are physical constants and z is the depth below the free surface, between the brine and the atmosphere. The pressure at the free surface is given by the constant p_0.

(iii) A square plate of side length 10 m is submerged in the brine so that the top edge of the plate lies in the free surface, and the faces of the plate are vertical. Find the magnitude of the total surface force on one of the faces of the plate. (Take $p_0 = 10^5$ Pa, $a = 2000\ \text{kg m}^{-3}$, $b = 12\ \text{kg m}^{-4}$ and the magnitude of the acceleration due to gravity as $10\ \text{m s}^{-2}$.) [7]

Question 2

A disc of diameter D immersed in a fluid of density ρ and coefficient of viscosity μ has a constant angular speed ω. The rate at which energy is produced to drive the disc is P. Use the method of dimensional analysis to show that one possible formula for P is

$$P = \rho\omega^3 D^5 f\left(\frac{\rho\omega D^5}{\mu}\right),$$

where f is an undetermined function. [6]

Question 3

(i) Determine whether or not the flow whose velocity vector field, in cylindrical polar coordinates, is given by

$$\mathbf{u} = \frac{m}{r}\mathbf{e}_r + \frac{k(t)}{r}\mathbf{e}_\theta \quad (r \neq 0),$$

is incompressible. (Here m is a constant and k is a function of time t.)

(ii) In Cartesian coordinates (x, y), the velocity vector field for the two-dimensional flow of an inviscid, incompressible fluid of density ρ and pressure p is

$$\mathbf{u} = -3y\mathbf{i} + (3x + 4t)\mathbf{j},$$

where t represents time, and the body force per unit mass is

$$\mathbf{F} = 4\mathbf{j}.$$

Give the x and y components of Euler's equation for this flow and use them to find the pressure distribution in the fluid (to within an arbitrary function of time). [7]

Question 4

(i) Determine the Fourier sine series for the function $f(x) = x$ on the interval $0 < x < 1$.

(ii) Sketch the graph, for $-2 \leq x \leq 2$, of the function defined by this Fourier sine series. [7]

Question 5

Consider the differential equation

$$(1 - x)\frac{dy}{dx} + y = 2x.$$

Determine its power series solution about $x = 0$ for which $y = 3$ when $x = 0$, giving the first three terms and showing that subsequent terms have coefficients given by

$$a_n = \frac{2}{n(n-1)} \quad (n \geq 3).$$ [7]

Question 6

(i) Show that if deep water wave theory applies, then the speed of groups of waves (the group velocity) equals half of the wave speed.

(ii) Consider a wave tank containing water of equilibrium depth 0.5 m. Pure sinusoidal waves of wavelength λ can be propagated in the water.

(a) Determine whether the deep wave approximation or the shallow wave approximation is appropriate, and use it to find the wave speed, in each of the cases:

(i) $\lambda = 0.05$ m,

(ii) $\lambda = 12$ m.

(b) If waves of wavelength $\lambda + \delta\lambda$ and $\lambda - \delta\lambda$ are propagated simultaneously, find the speeds of the groups of waves that are visible in the tank for the cases:

(i) $\lambda = 0.05$ m, $\delta\lambda = 0.001$ m,

(ii) $\lambda = 12$ m, $\delta\lambda = 0.001$ m.

(Take the magnitude of the acceleration due to gravity as $10\,\mathrm{m\,s^{-2}}$.) [6]

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The formula for P should be:

$$P = \rho\omega^3 D^5 f\left(\frac{\rho\omega D^2}{\mu}\right).$$

PART II

Answer **THREE** *questions in this part.*

Each question carries 20% of the total examination marks.

Question 7

(i) Write down Laplace's equation in spherical polar coordinates (r, θ, ϕ) and show that a separation of variables solution that is independent of the angle ϕ has r-dependence in the form of a Cauchy–Euler equation. [5]

(ii) Derive the general solution of the Cauchy–Euler equation

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - n(n+1)R = 0 \quad (r > 0),$$

where n is a non-negative integer. [3]

(iii) Show that the solution $u(r,\theta)$ of Laplace's equation in spherical polar coordinates in the region $r > a, 0 < \theta < \pi$ that has cylindrical symmetry about the polar axis, is bounded on the polar axis and satisfies

$$u(r,\theta) \to Ur\cos\theta \quad \text{as } r \to \infty,$$
$$\frac{\partial u}{\partial r}(a,\theta) = 0, \quad 0 \le \theta \le \pi,$$

is given by

$$u(r,\theta) = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta$$

to within an arbitrary constant. [12]

Question 8

(i) One form of Bernoulli's equation is

$$\frac{p}{\rho} + \frac{1}{2}u^2 - \Omega = \text{ constant along any curve drawn in the fluid.}$$

What assumptions have been made in deriving this formula? [2]

(ii) State Torricelli's formula, explaining clearly in words what the formula means and what any symbols represent.

Show how Torricelli's formula may be derived from an appropriate form of Bernoulli's equation, stating clearly any assumptions that are made. [8]

(iii) A vertical, cylindrical vessel has a waste pipe of length 10 mm and cross-sectional area $20\,\text{mm}^2$ protruding vertically downwards from its base. Water enters the vessel at the volume flow rate of $3 \times 10^{-5}\ \text{m}^3\,\text{s}^{-1}$. What is the depth of the water in the vessel in the steady state? State clearly any assumptions you make in carrying out the calculation. You may take g, the magnitude of the acceleration due to gravity, to be $10\,\text{m}\,\text{s}^{-2}$. [8]

(iv) Suppose water flows out of the waste pipe but the water in the vessel is *not* replenished, so that the speed of fall of the free surface (of the fluid) is non-zero. Are the assumptions you made in Part (iii) satisfied in this case? State, briefly, under what condition(s) the assumptions might be justified. [2]

Question 9

(i) An inviscid, incompressible fluid, influenced only by conservative forces, has a velocity vector field referred to cylindrical polar coordinates (r, θ, z) given by

$$\mathbf{u} = \alpha(r, t)\mathbf{k},$$

where $\alpha(r,t)$ is a known scalar function of r and time t, and $\mathbf{k}$ is a unit vector in the z-direction. The simple, closed curve C lies on a vortex tube in the fluid and encloses a plane cross-sectional surface S, of area A, of the tube.

(a) State Kelvin's Theorem. Is Kelvin's Theorem applicable to the situation described above? Justify your answer.

(b) Describe clearly a typical vortex tube of the flow and give its strength as an integral in terms of α, r and A.

(c) If $\alpha(r, 0) = 0$ show that the flow is always irrotational. [8]

(ii) A fluid of density ρ and coefficient of viscosity μ flows past a cylinder of radius a, placed transverse to the flow. The flow some distance from the cylinder has speed U.

(a) Show that each of the quantities

$$\frac{a}{U}, \quad \frac{a^2\rho}{\mu} \quad \text{and} \quad \frac{\mu}{\rho U^2}$$

has the dimensions of time. Define a Reynolds number for the flow.

(b) With reference to a diffusion time scale and a convection time scale (both of which you must give) and the Reynolds number for the flow, defined in Part (a), write three short paragraphs explaining the creation and transport of vorticity and discussing the competing effects of diffusion and convection. Use sketch diagrams to explain your answer if you wish.

(c) Explain why Kelvin's Theorem and the Persistence of Irrotational Motion are relevant concepts in the 'main stream' region, away from the immediate vicinity of the cylinder. [12]

Question 10

(i) State, *briefly*, what is meant by the *standard form* of a linear, second-order partial differential equation whose coefficients are, in general, functions of two independent variables. [2]

(ii) For each of (a) the wave equation, (b) the diffusion equation and (c) Laplace's equation, write down an appropriate form of the equation in Cartesian coordinates, classify it, and give its corresponding standard form. [6]

(iii) The change of variables

$$\zeta = 2y + x^2, \quad \phi = 2y - x^2$$

reduces the equation

$$x^2\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} - 4x^2\frac{\partial u}{\partial y} + \left(4x + \frac{1}{x}\right)\frac{\partial u}{\partial x} = 0 \quad (x \neq 0) \tag{1}$$

to the form

$$\frac{\partial^2 u}{\partial \zeta \partial \phi} - \frac{\partial u}{\partial \phi} = 0. \tag{2}$$

(You are *not* asked to derive Equation (2).)
Use Equation (2) to show that the general solution of Equation (1) is

$$u = g(x^2 + 2y) + e^{(x^2+2y)}h(2y - x^2)$$

and determine the particular solution that satisfies the additional conditions

$$u(1, y) = 1 + y^2,$$
$$\frac{\partial u}{\partial x}(1, y) = 2y.$$

[12]

Question 11

(i) Write down the one-dimensional wave equation for displacements u that depend on the coordinate x, time t and have wave speed c. Use the method of separation of variables to show that if there is a free-end boundary condition at $x = l$ and the displacement at $x = 0$ is zero, then the solution of this equation for $0 \leq x \leq l, t \geq 0$ has the form

$$u(x, t) = \sum_{n=0}^{\infty}\left(c_n \sin\frac{\left(n+\frac{1}{2}\right)\pi ct}{l} + d_n \cos\frac{\left(n+\frac{1}{2}\right)\pi ct}{l}\right)\sin\frac{\left(n+\frac{1}{2}\right)\pi x}{l}, \tag{1}$$

where c_n and d_n are constants. [14]

(ii) Further, if the displacement is subject to the initial conditions:

$$u(x, 0) = 0,$$
$$\frac{\partial u}{\partial t}(x, 0) = v\sin\left(\frac{\pi x}{2l}\right), \quad v \text{ constant},$$

show that Equation (1) reduces to a standing wave for $0 < x < l$. [6]

[END OF QUESTION PAPER]