Part I

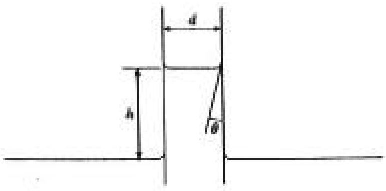
Answer **ALL SIX** *questions in this part.*

The questions in this part are not all worth the same number of marks.

The number of marks assigned to each question is given in square brackets.

Part I as a whole carries 40% of the total examination marks.

Question 1



When a thin tube is dipped into a reservoir of liquid, the free surface of the liquid rises, or falls, inside the tube. This phenomenon is caused by surface tension, defined as force per unit length. Assuming that the height h of the liquid in the tube above the reservoir level depends on the density ρ of the liquid, the surface tension S, the diameter d of the tube, the angle θ of contact between the liquid and the tube and the magnitude g of the acceleration due to gravity, use the method of dimensional analysis to show that

$$h = d\, f\left(\frac{g\rho d^2}{S}, \theta\right),$$

where f is an undetermined function of two variables. [6]

Question 2

Consider the velocity vector field

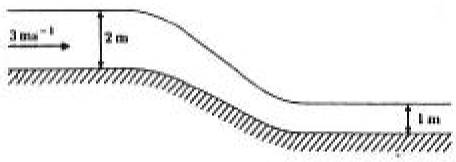
$$\mathbf{u}(r,\theta,z) = -r^{-\frac{1}{2}} \sin \tfrac{1}{2}\theta\, \mathbf{e}_r - r^{-\frac{1}{2}} \cos \tfrac{1}{2}\theta\, \mathbf{e}_\theta \qquad (r > 0, -\pi < \theta < \pi)$$

for the steady two-dimensional flow of an incompressible fluid with respect to cylindrical polar coordinates.

(i) Write down the equations satisfied by the stream function for the velocity field $\mathbf{u}$. Hence find the stream function $\psi(r,\theta)$ for this flow and the equation of the streamlines.

(ii) In particular, find the equation of the streamline which passes through the point $r = 1$, $\theta = 0$, $z = 0$. Hence make a rough sketch of this streamline, indicating the direction of motion. [6]

Question 3



Water is flowing in an open rectangular horizontal channel at a depth of 2 metres and a speed of 3 $\mathrm{m\,s^{-1}}$. It then flows smoothly down a chute into another open rectangular horizontal channel of the same width where the depth of water is 1 metre. Assuming the water can be modelled as an inviscid incompressible fluid and that the flow is steady and irrotational, find the speed of the water in the lower channel and the difference in height between the two channel floors. You may take the magnitude g of the acceleration due to gravity as 10 $\mathrm{m\,s^{-2}}$. [5]

Question 4

The function $u(x, y)$ satisfies the partial differential equation

$$x^2\frac{\partial^2 u}{\partial x^2} - 4xy\frac{\partial^2 u}{\partial x \partial y} + 4y^2\frac{\partial^2 u}{\partial y^2} + 6y\frac{\partial u}{\partial y} = 0 \qquad (x \neq 0).$$

The characteristic coordinates may be chosen to be

$$\zeta = x^2 y \quad \text{and} \quad \phi = x.$$

(i) Use the coordinates ζ and ϕ to transform the partial differential equation to standard form.

(ii) Hence find the general solution u of the partial differential equation as a function of x and y. [8]

Question 5

Determine the first three non-zero terms of the power series solution about $x = 0$ of the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - y = 0$$

which satisfies the initial conditions

$$y(0) = 0 \quad \text{and} \quad \frac{dy}{dx}(0) = 1.$$

State the interval of convergence for the power series solution. [8]

Question 6

The following equations are used in the linear theory of gravity water waves when the water is of finite equilibrium depth h:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad (0 < z < h),$$

$$\frac{\partial\phi}{\partial z} = 0 \qquad \text{at } z = 0,$$

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \quad \text{at } z = h,$$

where ϕ is the velocity potential and g is the magnitude of the acceleration due to gravity. In these equations the channel floor is $z = 0$ and the undisturbed free surface is $z = h$ with z measured vertically upwards.

By considering a solution of the form

$$\phi(x, z, t) = f(z)\cos(kx - \omega t),$$

show that

(i) the frequency ω is related to the wave number k by the dispersion relation

$$\omega^2 = gk \tanh kh;$$

(ii) the relationship between the wave speed c and wavelength λ is

$$c^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}.$$

[7]

PART II

Answer THREE *questions in this part.*

Each question carries 20% of the total examination marks.

Question 7

Consider the flow of a viscous fluid of constant density ρ and coefficient of viscosity μ which is bounded by two infinite parallel planes $z = 0$ and $z = h$. The plane $z = 0$ is stationary whereas the plane $z = h$ moves parallel to itself in the direction of the x-axis with velocity $Ue^{-\alpha t}\mathbf{i}$, where U and α are positive constants. There is no applied pressure gradient and the body force due to gravity may be neglected. The fluid flow is modelled by assuming that the velocity field $\mathbf{u}$ is independent of x and y and has no component in the y-direction.

(i) Briefly explain why the above modelling assumptions are reasonable. [2]

(ii) State the boundary conditions on the velocity function $\mathbf{u}$. [2]

(iii) Use the equation of continuity, together with the modelling assumptions and boundary conditions, to show that the velocity field is $\mathbf{u} = u_1\mathbf{i}$ where u_1 is a function of z and t. [4]

(iv) Further, use the Navier-Stokes equation to show that $u_1(z, t)$ satisfies the partial differential equation

$$\frac{\partial u_1}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2 u_1}{\partial z^2}.$$ [4]

(v) By assuming a solution of the form

$$u_1(z, t) = e^{-\alpha t} f(z),$$

find the velocity field which satisfies the boundary conditions from part (ii). [6]

(vi) Find the force per unit area on the fixed plane $z = 0$ at time t due to the fluid. [2]

Question 8

(i) Show that the Fourier sine series for the function

$$f(x) = \cos x$$

on the interval $0 < x < \pi$ is

$$\frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m \sin 2mx}{4m^2-1}.$$ [7]

(ii) Show that the eigenvalue problem

$$X''(x) + \lambda X(x) = 0 \qquad (0 < x < \pi),$$
$$X(0) = 0,$$
$$X(\pi) = 0$$

has eigenvalues $\lambda_n = n^2 (n = 1, 2, 3, \ldots)$ and corresponding eigenfunctions

$$X_n(x) = \sin nx.$$ [4]

(iii) The equation governing the temperature distribution $u(x,t)$ in an insulated bar of length π is given by the diffusion equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi, t > 0),$$

where x and t represent distance along the bar and time respectively and k is a positive constant. Initially the temperature distribution of the bar is

$$u(x, 0) = \cos x \qquad (0 < x < \pi).$$

At time $t = 0$ the two ends of the bar are immersed in blocks of ice and are thus maintained at 0 °C.

(a) Write down the boundary conditions at $x = 0$ and $x = \pi$ for the temperature distribution. [2]

(b) Use the method of separation of variables to determine the temperature distribution $u(x,t)$ at times $t > 0$. [7]

Question 9

Consider the regular Sturm–Liouville problem defined by the differential equation

$$((4x+1)^2u')' - u + \lambda u = 0 \qquad (0 < x < 1)$$

with the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 0.$$

(i) State the orthogonality condition which is satisfied by eigenfunctions of the problem. [1]

(ii) Find upper and lower bounds for each eigenvalue λ_n $(n = 1, 2, 3, \ldots)$ of the problem and hence show that all the eigenvalues are greater than 5. [8]

(iii) By using the change of variable

$$x = \tfrac{1}{4}(t - 1),$$

find the general solution of the differential equation in the case $\lambda > 5$ and hence solve the eigenvalue problem. [11]

Question 10

(i) Verify that

$$\phi(r,\theta,z) = \left(Ar + \frac{B}{r}\right)\cos\theta + C\theta$$

could be the scalar potential field for the irrotational flow of an inviscid incompressible fluid where r, θ, z are cylindrical polar coordinates and A, B, C are constants. [5]

(ii) An inviscid incompressible fluid flows irrotationally past an infinite solid cylinder whose surface is $r = a$. At a large distance from the cylinder the velocity of the fluid is $\mathbf{u} = U\mathbf{i}$, where U is a constant. In addition there is a circulation $-\kappa$ around the cylinder. Show that the scalar potential field for this problem is

$$\phi(r,\theta,z) = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\kappa}{2\pi}\theta.$$ [8]

(iii) Ignoring the effects of gravity, obtain an expression for the pressure distribution on the surface of the cylinder. Hence show that the total force per unit length on the cylinder is

$$\mathbf{F} = \rho U \kappa \mathbf{j},$$

where ρ is the density of the fluid. [7]

Question 11

(a) (i) Discuss briefly the problems of measuring the coefficient of viscosity of a liquid.

(ii) Describe briefly how a falling-sphere viscometer is used and give one disadvantage of this type of viscometer.

(iii) Describe in a few words the use of a viscometer which involves the flow of the liquid through a pipe. [8]

(b)

(i) (ii)

The figure shows an aerofoil facing a streaming fluid at (i) low incidence and (ii) high incidence. With the aid of suitable diagrams contrast the flow patterns for a viscous model of these fluid flows. Comment on the difference between the lift forces in each case and suggest what might happen to an aeroplane with such a high angle of incidence as that in (ii). How might aircraft designers alleviate the problem of boundary-layer separation in situations such as (ii)? [6]

(c) Transverse waves on a light elastic string can be modelled by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

Briefly describe the initial and boundary conditions which are appropriate for the solution of this partial differential equation for

(i) an infinite string,

(ii) a finite string.

State in one or two sentences methods of solution which can be used for

(i) an infinite string,

(ii) a finite string.

Describe in a few words what is meant by

(i) a travelling or progressive wave,

(ii) a standing wave. [6]

[END OF QUESTION PAPER]