MST207/T

Second Level Course Examination 1999

Mathematical Methods, Models and Modelling

Thursday, 21 October, 1999 10.00 am – 1.00 pm

Time allowed: 3 hours

There are twenty questions on this paper. Questions 1–14 are each worth 5 marks; questions 15–20 are each worth 15 marks. For your convenience, the shorter questions have been grouped together as Part I of the paper, the longer questions as Part II. The marks allocated to each part of each question in Part II of the paper are given in square brackets in the margin. In each part of the paper the questions are arranged, as far as possible, in unit order.

You should do as many questions as you can. All of the answers that you submit will be marked. You are not expected to answer all of the questions: full marks may be obtained by, for example, correctly answering 11 questions from Part I and 3 questions from Part II. The maximum credit for the paper is 100 marks; scores greater than 100 will be rounded down to 100.

Unless you are directed otherwise in the question, you may use any formula or other information from the Handbook in your answers.

Write your answers in the answer book(s) provided.

At the end of the examination, **check that you have written your personal identifier and examination number on each answer book used.** Failure to do so will mean that your work cannot be identified. Attach all your answer books together using the fastener provided.

PART I

Each question in this part of the paper is worth 5 marks.

Question 1

Solve the initial-value problem

2 $$\frac{dy}{dx} = 2x(1+y^2), \quad y(0) = 1,$$

expressing your answer in the form $y = f(x)$.

Question 2

Vectors $\mathbf{a}$ and $\mathbf{b}$ are defined in terms of Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ by

$$\mathbf{a} = -\mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

(i) Show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

4 (ii) Find three vectors $\widehat{\mathbf{a}}$, $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{c}}$ such that

- $\widehat{\mathbf{a}}$ is in the same direction as $\mathbf{a}$, and $\widehat{\mathbf{b}}$ is in the same direction as $\mathbf{b}$;
- $\widehat{\mathbf{a}}$, $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{c}}$ are mutually perpendicular;
- $\widehat{\mathbf{a}}$, $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{c}}$ are unit vectors.

Question 3

A uniform ladder stands on rough horizontal ground and leans against a vertical wall, which can be taken to be smooth. The angle between the ladder and the ground at its base is $\pi/3$. The mass of the ladder is 150 kg, and its length is 8 metres. The coefficient of static friction between the ground and the ladder is $\frac{1}{2}$.

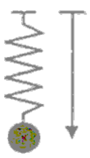
5

Determine whether or not a person whose mass is half that of the ladder can stand at the very top of the ladder without it slipping.

Question 4

A model spring is suspended from a fixed point. Particles of different masses are attached to its lower end in turn, and the equilibrium length of the spring is measured in each case.

7



When a particle of mass 0.5 kg is attached to the spring, its length is 1.25 metres. When this particle is replaced by a particle of mass 1 kg, the spring's length is 1.5 metres.

(i) Find the stiffness and natural length of the spring, giving the appropriate units of each of your answers. (The magnitude of the acceleration due to gravity is $9.81\,\mathrm{m\,s^{-2}}$.)

(ii) A particle of mass 5 kg is attached to the spring, and the system is set in motion. What will be the period of the oscillations of the particle? You may assume that no forces act other than the spring force and the weight of the particle.

Question 5

A tennis player, attempting to return service, strikes the ball so that it leaves her racket with a speed of $20\,\mathrm{m\,s^{-1}}$, at an angle of $10°$ ($\pi/18$ radians) above the horizontal. The horizontal distance from the point at which the ball is struck to the net, in the direction in which the ball is travelling, is 12 metres; the height of the point at which the ball is struck above the ground is 0.75 metres; the height of the top of the net above the ground is 1 metre.

14

Determine whether or not the ball clears the net.

You may treat the ball as a particle, and assume that no forces act on it other than the force of gravity. (The magnitude of the acceleration due to gravity is $9.81\,\mathrm{m\,s^{-2}}$.)

Question 6

Use the matrix form of Gaussian elimination to find the solution of the following set of simultaneous equations.

9

$$\begin{cases} x_1 + 3x_2 - 2x_3 = 2 \\ 2x_1 + 2x_2 = 3 \\ -2x_1 - 2x_2 + 5x_3 = 2 \end{cases}$$

Question 7

The eigenvalues of the matrix

10

$$\begin{bmatrix} 18 & -10 \\ 30 & -17 \end{bmatrix}$$

are -2 and 3. (You are not asked to prove that this is so.)

Using this information, find the general solution of the following system of differential equations.

$$\begin{cases} \dot{x} = 18x - 10y \\ \dot{y} = 30x - 17y \end{cases}$$

Question 8

A particle of mass m moves on a smooth horizontal surface. It is attached to a fixed point P by a model spring of natural length l_0 and stiffness k, and to another fixed point Q by a model damper of damping constant r. The particle moves along the straight line PQ, between the points P and Q.

17

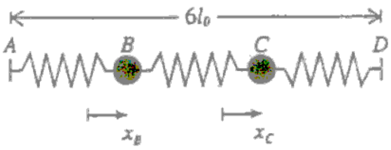
(i) Derive the equation of motion of the particle.

(ii) Suppose that (in S.I. units) $m = 2$, $k = 3$ and $r = 5$. Will the particle's motion be oscillatory? Justify your answer.

Question 9

18

Three model springs AB, BC, CD each have natural length l_0; their stiffnesses are k, $2k$ and $3k$ respectively. A particle of mass $2m$ is attached to the springs at B, and another, of mass m, is attached to the springs at C. The ends A and D are fixed to two points a horizontal distance $6l_0$ apart. The system is free to move along the horizontal line AD, as shown in the figure.

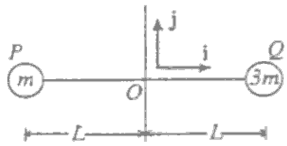


Find the differential equations satisfied by the displacements x_B and x_C of the particles B and C from their equilibrium positions.

Question 10

19

Two particles, of masses m and $3m$ respectively, lie on a smooth horizontal surface. They are placed at points P and Q a distance $2L$ apart, as shown in the figure.



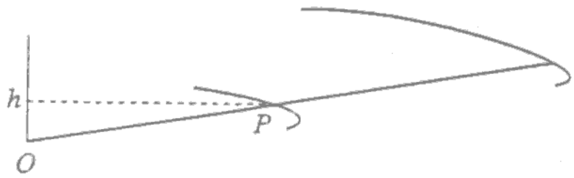
(i) Find the position of the centre of mass of the two particles, giving your answer as a vector relative to the origin O and Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ shown in the figure.

(ii) The particles are set in motion, at the same instant, with the following velocities: the velocity of the particle at P is $-3v\,\mathbf{j}$, and that of the particle at Q is $v\,\mathbf{j}$, where v is a constant. No forces act on the particles in horizontal directions. Describe the motion of the centre of mass of the two particles.

(iii) Suppose that, instead of being independent of each other, the particles are connected by a model spring along PQ. The particles are set in motion as before, and as they move the spring remains straight. Describe the motion of their centre of mass in this case.

Question 11

A cycle-racing track consists of a portion of a cone, as shown in the figure. A cyclist is riding at constant speed v round a horizontal circle on the track at a height h above the apex of the cone. The cyclist and bicycle may be modelled as a particle.



In the diagram, O is the apex of the cone and P the (instantaneous) position of the cyclist. By considering the cyclist's equation of motion relative to the vertical plane through OP, show that there is a certain speed at which the cyclist can go such that that there will be no component of frictional force perpendicular to the direction of motion of the bicycle.

Question 12

Find the second-order Taylor polynomial about $(-1, 1)$ of the function

12

$$f(x, y) = \frac{x + 2}{y + 1}.$$

Question 13

This question is concerned with the numerical solution of the differential equation

$$\frac{dy}{dx} = x^2 + \cos(y).$$

Let $y(x)$ be the particular solution of this equation which satisfies the condition

$$y(0.2) = 0.5.$$

A computer was used to calculate an approximation, $Y(0.3)$, for $y(0.3)$ by the Euler-trapezoidal method with two different step sizes, and the following results were obtained.

step size	$Y(0.3)$
0.05	0.591 798
0.02	0.591 778

Estimate the largest step size that would give the value of $y(0.3)$ accurate to 6 decimal places, when used in the Euler-trapezoidal method. State any assumptions that you make in obtaining your answer.

Question 14

Consider the function

$$f(x, y, z) = 3x^2y^3 + 5yz^2 + xyz.$$

(i) Find the rate of change of the function at the point $(0, 2, 1)$ in the direction determined by the unit vector

12

$$\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).$$

(ii) Let $\mathbf{F} = \mathbf{grad}\, f$. Find $\operatorname{div} \mathbf{F}$.

PART II

Each question in this part of the paper is worth 15 marks.

Question 15

Find the solution of the differential equation

3

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 15y = 4e^{-3x} - 5\cos(5x) - 3\sin(5x)$$

that satisfies the initial conditions

$$y = \tfrac{1}{10} \text{ and } \frac{dy}{dx} = 2 \text{ when } x = 0.$$ [15]

Question 16

6

When a rubber ball (or any other buoyant object) is moving in a vertical direction under water, it experiences two forces in addition to the force of gravity: a drag force opposing the motion, whose magnitude is proportional to the square of the ball's speed; and the buoyancy force, which is modelled as a constant upwards force. For convenience, denote the magnitude of the drag force by mkv^2 and that of the buoyancy force by mb, where m is the ball's mass, v its speed, and k and b are constants.

(i) What are the dimensions of k and b? Give your answers in terms of the base dimensions M, L and T. [2]

(ii) Derive the equation of motion of a rubber ball, submerged in water, moving vertically in a straight line. Use an upward-pointing coordinate axis. Consider carefully whether one equation is enough to cover both upward and downward motion, or whether two different equations are required. [5]

(iii) Set $b - g = c$, and suppose that $c > 0$. Describe the motion of a rubber ball which is released from rest at the bottom of a deep column of water. [3]

(iv) A rubber ball is held at the bottom of a swimming pool of depth d, and then released from rest. Assuming that $c > 0$, find the ball's speed when it reaches the surface of the water, giving your answer as an expression involving c, k and d. [5]

Question 17

A domestic hot-water tank consists of a copper cylinder of internal radius 0.75 metres and height 1 metre. The thickness of the copper sheet from which it is made is 5 mm. The tank is unlagged. It stands on a concrete base, so no heat can escape from its bottom, but heat can escape from its top as well as from its curved surface. The tank is full of water, which can be heated and kept hot by an immersion heater. Heat is transferred from the water in the tank to the surrounding air by convection at the surfaces between the water and the tank, by conduction through the sides and top of the tank, and by convection at the surfaces between the tank and the air.

The convective heat transfer coefficient for convection at the surfaces between the water and the tank is $1000\,\mathrm{W\,m^{-2}\,K^{-1}}$; the convective heat transfer coefficient for convection at the surfaces between the tank and the air is $10\,\mathrm{W\,m^{-2}\,K^{-1}}$; and the thermal conductivity of copper is $385\,\mathrm{W\,m^{-1}\,K^{-1}}$. The density of water is $998\,\mathrm{kg\,m^{-3}}$ and its specific heat capacity is $4190\,\mathrm{J\,kg^{-1}\,K^{-1}}$.

(i) Show that the rate of heat transfer from the water to the air, q, is given by

$$q = \lambda(\Theta - \Theta_a),$$

where Θ is the water temperature, Θ_a is the air temperature, and λ is a constant. Find the numerical value of λ, and state its units. [5]

(ii) Find at what constant rate energy has to be supplied by the heater if the water temperature is to be maintained at 80 °C when the air temperature is 20 °C. [2]

(iii) The heater is switched off. The temperature of the water in the tank is 80 °C, the temperature of the air is 20 °C, as before. How long will it be before the water temperature drops to 60 °C? [8]

Question 18

Find all of the equilibrium points of the system of differential equations

$$\begin{cases} \dot{x} = -(x-1)(y+2) \\ \dot{y} = 2x + y - 4. \end{cases}$$

For each equilibrium point, say whether it is stable or unstable, and draw a sketch of the vector field corresponding to the system in the neighbourhood of the equilibrium point which illustrates the nature of the equilibrium point. [15]

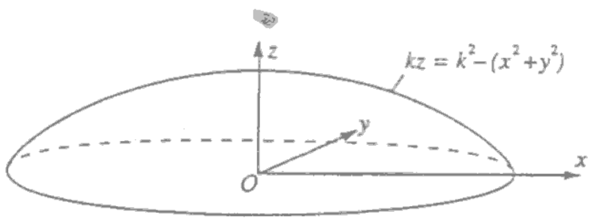
Question 19

(a) Find the area integral of the function

$$f(x, y) = x^2 y$$

over the region of the (x, y)-plane enclosed by the straight line $y = x$ and the parabola $y = x^2$. [5]

(b) A body of uniform density D is bounded by the paraboloid $kz = k^2 - (x^2 + y^2)$ and the (x, y)-plane, where k is a positive number.

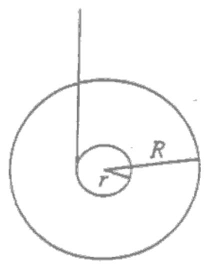


(i) Find the mass, M, of the body. [5]

(ii) Show that the moment of inertia of the body about the z-axis is $\frac{1}{3}Mk^2$. [5]

Question 20

A yo-yo consists of two solid uniform circular discs, each of mass M and radius R, attached to the two ends of a solid uniform cylindrical axle of mass m and radius r, where $R > r$. A string is wound round the axle; one end of the string is held fixed, and as the yo-yo moves the string is wound off the axle, causing the yo-yo to rotate. Consider the vertical downwards motion of the yo-yo, with the string vertical. You may assume that the yo-yo has no sideways motion.



(i) Find the moment of inertia I of the yo-yo about its axis, in terms of M, m, R and r. [5]

(ii) Write down a relationship between θ, the angle through which the yo-yo turns about its axis, and x, the distance that the centre of the yo-yo moves downwards. [2]

(iii) Show that the magnitude of the yo-yo's downwards acceleration is

$$\left(\frac{2(2M+m)r^2}{2M(R^2+2r^2)+3mr^2}\right) g,$$

where g is the magnitude of the acceleration due to gravity. [8]

[END OF QUESTION PAPER]