Question 16

(a) $2160 = 2^4 \times 3^3 \times 5$. 1

prime power	factors				label
5				5	$5a$
3^3				3^3	$3a$
			3	3^2	$3b$
		3	3	3	$3c$
2^4				2^4	$2a$
			2	2^3	$2b$
			2^2	2^2	$2c$
		2	2	2^2	$2d$
	2	2	2	2	$2e$

The number of Abelian groups of order 2160 is $1 \times 3 \times 5 = 15$. 4

(b) The extra condition given means that the canonical decomposition of such a group cannot contain a cyclic group of order greater than 60.

(i) The last torsion coefficient cannot be greater than 60, so (in the table above) $3a$ and $3b$ are both excluded as they lead to a torsion coefficient of at least 90.
For the same reason, $2a$ and $2b$ are ruled out. Thus, we can only have $5a$ and $3c$, together with $2c$–$2e$, three possibilities in all. 3

(ii) $2c$ gives $\mathbb{Z}_3 \times \mathbb{Z}_{12} \times \mathbb{Z}_{60}$.
$2d$ gives $\mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_{60}$.
$2e$ gives $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_{30}$. 3 1 each

(iii) For the three groups listed in part (b)(ii), the 2-primary components are

$$\mathbb{Z}_4 \times \mathbb{Z}_4,$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4,$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

respectively.

Since $\mathbb{Z}_4$ has a cyclic subgroup of order 2, the first has a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the Klein group. The other two clearly have such a subgroup.
The first two have $\mathbb{Z}_4$ subgroups. The last has all its elements of order 2, so has no such subgroup. 4