Part IIB

Do not attempt more than two questions from Part IIB.

Question 15

Let H be a subgroup of a group G. The *normalizer*, L, of H in G is defined by

$$L = \{x \in G : xH = Hx\}.$$

(a) Prove that L is a subgroup of G and that H is a normal subgroup of L. [7]

(b) Prove that if K is any subgroup of G containing H such that H is a normal subgroup of K, then K is a subgroup of L. [3]

(c) Find the normalizers of the following two subgroups of the permutation group S_3:

(i) $\{e, (1\,2)\}$;

(ii) $\{e, (1\,2\,3), (1\,3\,2)\}$.

(Give a brief reason for each answer.) [5]

Question 16

(a) Find all Abelian groups of order 1 575 (up to isomorphism), expressing each of them *both* in canonical form *and* in p-primary form. [9]

(b) Write down the groups that you found in part (a) which possess a subgroup isomorphic to $\mathbb{Z}_{45}$, justifying your answer. [6]

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Question 17

Let G be a group of order 72.

(a) Show that G has either 1 or 4 Sylow 3-subgroups. [3]

(b) Consider the case where G has 4 Sylow 3-subgroups X_1, X_2, X_3 and X_4.

Using the alternative definition of group action given in Lemma 5.1 of *Unit IB2*, show that conjugation by elements of G on the elements of the set

$$X = \{X_1, X_2, X_3, X_4\}$$

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produces an action of the group G on the set X and hence a homomorphism ϕ from G to the permutation group S_4. [6]

(c) Prove that kernel of the homomorphism ϕ of part (b) is neither G nor $\{e\}$. [4]

(d) Use parts (a) and (c) to show that no group of order 72 is simple. [2]

[END OF QUESTION PAPER]