Solutions to Part IIB

A reminder: at most two of your three questions attempted for Part II may be chosen from Part IIB.

Question 15

(a) By closure in G, G_1G_2 is a subset of G. 1

Closure
Let $g_1, h_1 \in G_1$, $g_2, h_2 \in G_2$. Then

$$g_1g_2, h_1h_2 \in G_1G_2$$

and

$$\begin{aligned}(g_1g_2)(h_1h_2) &= g_1(g_2h_1)h_2 \\ &= g_1(h_1g_2)h_2 \quad \text{(since all elements of } G_1 \text{ commute with all elements of } G_2\text{)} \\ &= (g_1h_1)(g_2h_2) \\ &\in G_1G_2.\end{aligned}$$ 2

Identity
As $e \in G_1$ and $e \in G_2, e = ee \in G_1G_2$. 1

Inverses
Let $g_1 \in G_1$, $g_2 \in G_2$. Then

$$\begin{aligned}(g_1g_2)^{-1} &= g_2^{-1}g_1^{-1} \\ &= g_1^{-1}g_2^{-1} \\ &\in G_1G_2.\end{aligned}$$ 1

(b) Deal with the morphism property, then the onto property.
Let $g_1, h_1 \in G_1$, $g_2, h_2 \in G_2$. Then

$$\begin{aligned}\phi((g_1, g_2)(h_1, h_2)) &= \phi((g_1h_1, g_2h_2)) \\ &= (g_1h_1)(g_2h_2) \\ &= g_1g_2h_1h_2 \\ &= \phi((g_1, g_2))\phi((h_1, h_2)).\end{aligned}$$

So, ϕ is a homomorphism. 3

Let $g_1g_2 \in G_1G_2$, where $g_1 \in G_1$, $g_2 \in G_2$. Then

$$\phi((g_1, g_2)) = g_1g_2$$

and ϕ is onto. 1

(c) Let $(g_1, g_2) \in \operatorname{Ker}(\phi)$, that is

$$\phi((g_1, g_2)) = g_1g_2 = e.$$

Thus $g_1 = g_2^{-1} \in G_2$ and $g_2 = g_1^{-1} \in G_1$.

Hence, $g_1, g_2 \in G_1 \cap G_2$. 3

(d) If $G_1 \cap G_2 = \{e\}$ then, by part (c),

$$(g_1, g_2) \in \operatorname{Ker}(\phi) \quad \text{implies} \quad g_1 = g_2 = e.$$

Hence, $\operatorname{Ker}(\phi) = \{e\}$ and ϕ is an isomorphism. 3