Part IIB

Do not attempt more than two questions from Part IIB.

Question 15

Let G be a finite group and let G_1 and G_2 be subgroups of G such that

$$g_1g_2 = g_2g_1 \quad \text{for all} \quad g_1 \in G_1, g_2 \in G_2.$$

(a) Show that

$$G_1G_2 = \{g_1g_2 : g_1 \in G_1, g_2 \in G_2\}$$

is a subgroup of G. [5]

(b) Let ϕ be the mapping from the direct product $G_1 \times G_2$ to G_1G_2 defined by

$$\phi((g_1, g_2)) = g_1g_2 \quad \text{for all} \quad g_1 \in G_1, g_2 \in G_2.$$

Show that ϕ is a homomorphism from $G_1 \times G_2$ *onto* G_1G_2. [4]

(c) Show that

$$(g_1, g_2) \in \text{Ker}(\phi) \quad \text{implies} \quad g_1, g_2 \in G_1 \cap G_2.$$ [3]

(d) Show that if $G_1 \cap G_2 = \{e\}$, then ϕ is an isomorphism. [3]

Question 16

(a) How many Abelian groups of order 2160 are there (up to isomorphism)? [5]

(b) An Abelian group has order 2160 and all its elements have orders not exceeding 60.

(i) How many such groups are there (up to isomorphism)? [3]

(ii) Express each possibility as a direct product of cyclic groups in canonical form. [3]

(iii) Show that any such group *must* have a subgroup isomorphic to the Klein group but *may or may not* have a cyclic subgroup of order 4. [4]

Question 17

(a) List the possibilities for the type and number of the Sylow subgroups of any group of order 36. [4]

(b) Which of the Sylow structures found in part (a) does the group D_{18} possess? [4]

(c) Let G be a group of order 36 which contains more than one Sylow 3-subgroup. By considering the action of G by conjugation on the set of Sylow 3-subgroups, show that G has a proper normal subgroup of order at least 3. [4]

(d) Show that no group of order 36 is simple. [3]

[END OF QUESTION PAPER]