M336/K

The Open University

Third Level Course Examination 1999

Groups and Geometry

Friday, 15th October, 1999 2.30 pm – 5.30 pm

Time allowed: 3 hours

There are **TWO** parts to this examination. You should attempt **BOTH**.

You may attempt **ALL** the questions in Part I. You should answer not more than **THREE** questions from Part II, including not more than **TWO** questions from Part IIA and not more than **TWO** questions from Part IIB.

You are advised to spend about 90 minutes on Part I and 80 minutes on Part II, and to leave yourself about 10 minutes for checking. Part I carries 55% of the total marks while Part II carries 45% of the total marks.

You are advised to show all your working and to give reasons for all your answers unless a question is explicitly phrased 'Write down...'. You should begin each answer on a new page of the answer book.

Tracing paper is available from the invigilator if you should require it.

Note: a Figure Sheet is provided with this question paper for use in answering Questions 11 and 13.

At the end of the examination

Check that you have written your examination number and personal identifier on each answer book used and on any separate sheets used, *particularly the Figure Sheet.* **Failure to do so may mean that your paper cannot be identified.** Attach all your answer books and the Figure Sheet together, using the fastener provided.

Write the numbers of the questions that you have attempted in the spaces provided on the front of the answer book.

Part I

You may attempt **ALL** *the questions in this part and are advised to spend about* **90 minutes** *on it.*

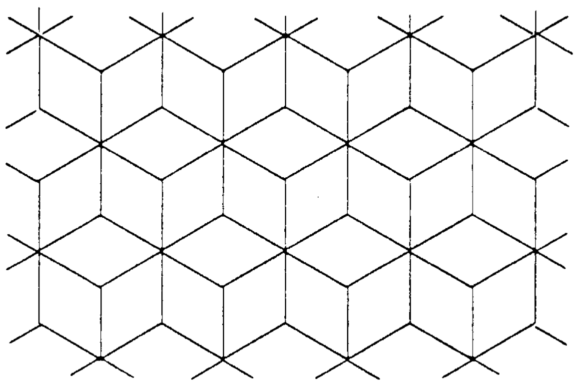
The marks for each question are given beside the question.

Question 1

(a) Find the inverse of the affine transformation $f = t[\mathbf{p}]\lambda[\mathbf{A}]$, where $\mathbf{p} = (1, -1)$ and

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix}.$$ [3]

(b) Consider the Laves tiling $\mathcal{T} = [3, 6, 3, 6]$, part of which is shown below.



$\mathcal{T}$

(i) Write down the number of non-congruent tile shapes that are produced if $\mathcal{T}$ is subjected to the affine transformation

$$(x, y) \mapsto (2x, y).$$ [1]

(ii) Write down the *maximum* number of non-congruent tile shapes that can be produced from $\mathcal{T}$ by an affine transformation. [1]

Question 2

The dihedral group D_6 is given by the presentation

$$D_6 = \langle r, s : r^6 = e, s^2 = e, sr = r^5 s \rangle$$

and the elements are written in standard form as

$$r^m \text{ and } r^m s, \qquad m = 0, \ldots, 5.$$

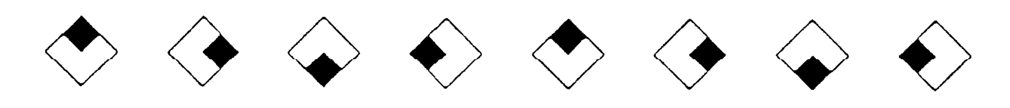
(a) Express the element $(r^4 s)(r^3 s)$ in standard form. [2]

(b) Express $(r^4 s)^{-1}$ in standard form. [2]

(c) Show that the dihedral group D_6 is non-Abelian. [1]

Question 3

(a) The following frieze, $\mathcal{F}$, is formed from a motif consisting of a square one-quarter of which is shaded. Copies of this motif are repeated at regular intervals, each rotated by a clockwise quarter-turn with respect to the motif to its left.



Use the algorithm of *Unit IB3* to determine the type of $\mathcal{F}$, making clear which questions you ask and what the corresponding answers are. [3]

(b) The Handbook gives the following standard form for the symmetry group of a frieze of Type 4.

$$\Gamma(F_4) = \{x, xg_{\frac{1}{2}} : x \in T_1; g_{1/2}^2 = t, g_{\frac{1}{2}}t = tg_{\frac{1}{2}}\}.$$

Explain why $\Gamma(F_4)$ requires only one generator, and write down such a generator. [2]

Question 4

Let G be a group, and let H and J be subgroups of G.

(a) Give an example to show that the set

$$HJ = \{hj : h \in H, j \in J\}$$

need not be a subgroup of G. [2]

(b) Prove that, if $HJ = JH$, then HJ is a subgroup of G. [3]

Question 5

(a) Square car stickers for a mathematical conference are made of transparent plastic, that can be stuck onto the windscreen using either surface. (Thus, there are eight ways to place a sticker on a windscreen so that the sides are vertical and horizontal.) Each sticker is divided by its diagonals into four triangles. Find the cycle index of the corresponding symmetry group acting on the configuration of four triangles. [3]

(b) Three colours (red, yellow and blue) are available for colouring the triangles; moreover, the intention is that each sticker issued should bear *at least* two colours. (Every triangle must be given one of the colours.) Find the number of distinguishable car stickers that can be made to this specification. [2]

Question 6

(a) Use the Euclidean Algorithm to find hcf$\{245, 231\}$. [2]

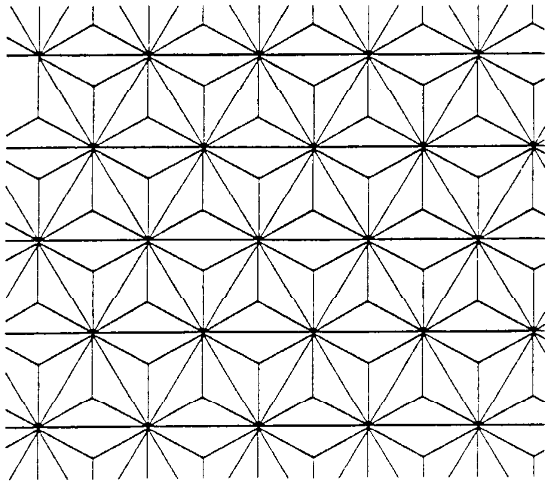
N.B. You *must* use the algorithm; *no marks* will be awarded for factorizing the two numbers.

(b) Use your answer to part (a) to express hcf$\{245, 231\}$ as an integer combination of 245 and 231. [2]

(c) Find lcm$\{245, 231\}$. [1]

Question 7

Let $\mathcal{T}$ be the Laves tiling $[3, 12, 12]$, a portion of which is shown below:



$\mathcal{T}$

(a) Write down $n_v(\mathcal{T})$ and $n_t(\mathcal{T})$. [2]

(b) Draw the tile-vertex diagram of $\mathcal{T}$. [2]

(c) Using a theorem from *Unit GE2*, or otherwise, find $n_e(\mathcal{T})$. [1]

Question 8

The finitely presented Abelian group A is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 5 & 5 \\ 6 & 0 & 12 \end{bmatrix}.$$

(a) Use the Reduction Algorithm to reduce this matrix to diagonal form. [3]

(b) Hence express A as a product of non-trivial cyclic groups in canonical form. [1]

(c) Write down a diagonal matrix, different from your answer to part (b), which also represents the group A. [1]

Question 9

Let

$$\mathbf{a} = (-1, 0), \quad \mathbf{b} = (1/2, \sqrt{3}/2).$$

(a) Write down the lattice type of $L(\mathbf{a}, \mathbf{b})$. (You need not give any justification.) [1]

(b) Show that $L(\mathbf{a}, 3\mathbf{b}) = L(3\mathbf{b} + 2\mathbf{a}, 3\mathbf{b} + \mathbf{a})$, and hence determine the lattice type of $L(\mathbf{a}, 3\mathbf{b})$. [3]

(c) Write down the lattice type of $L(\mathbf{a}, 2\mathbf{b})$. (You need not give any justification.) [1]

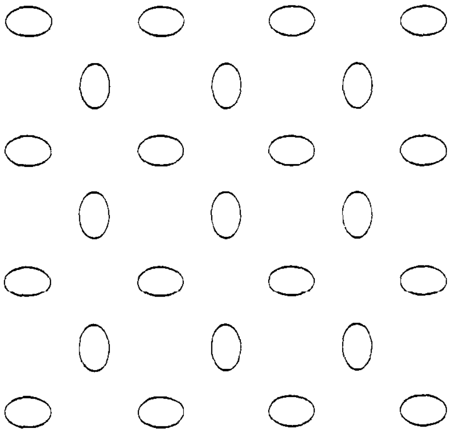
Question 10

(a) A non-Abelian group G has order 30. Prove that the centre of G has order at most 5. [3]

(b) Explain why the centre of the group $S_3 \times \mathbb{Z}_5$ has order 5. [2]

Question 11

Consider the wallpaper pattern W below, an enlarged copy of which is reproduced as Figure 1 on the Figure Sheet. (Note that the centres of the ellipses form a square lattice.)



W

(a) On Figure 1 of the Figure Sheet, draw a basic rectangle of W and shade in a generating region. [3]

(b) Write down the wallpaper type of W. (You need not give a justification.) [2]

Part II

You may attempt not more than THREE *questions from this part, and you are advised to spend about* 80 **minutes** *on it.*

You may choose not more than TWO *of your three questions from Part IIA and not more than* TWO *from Part IIB.*

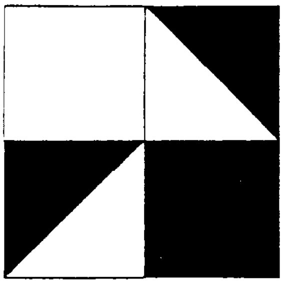
Each question carries 15% of the marks for the whole examination, and an indication of the allocation of marks within each question is given beside the question.

Part IIA

Do not attempt more than two questions from Part IIA.

Question 12

This question concerns ways of colouring a square S, drawn on one side of a piece of paper and partitioned into four quadrants. Each quadrant is then to be partitioned by a diagonal into two right-angled triangles; the triangles are each to be white or black. If both triangles in a quadrant are the same colour, then the partition is invisible. (An example of such a coloured square is illustrated below.)



(a) Show that, *without* taking any symmetry into account, there are 1296 ways to colour a square subject to these specifications. [4]

(b) Assume that such a square is drawn on a piece of paper that is itself square, so that it can be presented to an observer in four orientations. By using the Counting Lemma (page 27 of your Handbook), calculate how many inequivalent configurations can be constructed according to the above specification. [3]

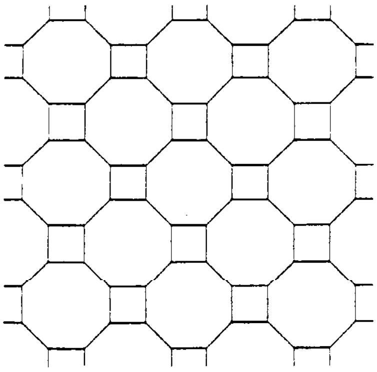
(c) Now assume that, instead of being drawn on paper, these squares are to be put together out of right-angled triangles, each being the same colour on each side; thus they can be turned over.

Use the Counting Lemma again, to calculate how many inequivalent configurations can be constructed in this case. [4]

(d) An alternative problem would concern a square divided into four quadrants each of which could be any one of six colours. Explain why this problem would have the same solution as part (b) above if the square could not be turned over, but would *not* have the same solution as part (c) above if the square *could* be turned over (having the same colours on the other side). [4]

Question 13

Consider the Archimedean tiling (4,8,8) as a wallpaper pattern, W; part of this pattern is shown below and enlarged copies are printed as Figures 2 and 3 on the Figure Sheet.



W

(a) How many orbits of rotation centres *of each order* are there under the action of the symmetry group $\Gamma(W)$? Clearly mark *one* representative of each such orbit, using Figure 2 of the Figure Sheet. Centres of order 2 (if any) should be marked with a small cross; of order 4 (if any), with a small square; of order 8 (if any), with a small circle. [4]

(b) On Figure 3 of the Figure Sheet, draw *one* representative of each orbit of *reflection* axes and *one* representative of each orbit of *glide* axes, under the action of the *translation* group, $\Delta(W)$. Use solid lines for the reflection axes and dashed or dotted lines for the glide axes. [3]

(c) Under the action of the *symmetry* group, $\Gamma(W)$, how many orbits are there of reflection axes, and how many of glide axes? [2]

(d) Carry out the algorithm described in the Handbook to determine the type of the pattern W. Your solution should make it clear which questions you ask and what the corresponding answers are. [2]

(e) Write down what the pattern type would be if:

(i) the *horizontal* rows of squares were alternately coloured black and white (that is, there were rows of black squares alternating with rows of white squares);

(ii) the *diagonal* rows of squares were alternately coloured black and white (that is, there were diagonal rows, from bottom left to top right, of black squares, alternating with similar rows of white squares). [4]

Question 14

(a) Let $L = L(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be a Bravais lattice, where

$$\mathbf{a} = (-2, 0, 0), \quad \mathbf{b} = (1, 1, 1), \quad \mathbf{c} = (0, 0, 2).$$

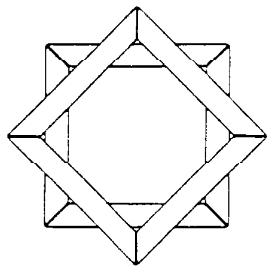
(i) Explain how Theorem 1.2 (page 55 of the Handbook) may be used to express L as $L(\mathbf{a}', \mathbf{b}', \mathbf{c}')$, where $L(\mathbf{a}', \mathbf{b}')$ is a square lattice. [3]

(ii) Use the result of part (a)(i) to find the type of L. [3]

(b) Find the Bravais lattice type of $L(\mathbf{d}, \mathbf{e}, \mathbf{f})$, where

$$\mathbf{d} = (3, 4, 5), \quad \mathbf{e} = (3, 4, -5), \quad \mathbf{f} = (4\sqrt{2}, -3\sqrt{2}, 0).$$ [3]

(c) A decoration is made from two identical squares, each made of four copper bars. The bars are welded together, in such a way that the centres of the squares are on an axis of symmetry and one square is rotated by 45 degrees with respect to the other. Let the direct symmetry group of the decoration be denoted by G^+ and its full symmetry group by G.



(i) G^+ clearly contains rotations about the symmetry axis of the decoration. If there are no other direct symmetries, explain why not; if there are, give a geometric description of them. [3]

(ii) Use Result 5.1 on pages 52 and 53 of the Handbook, to classify G. [3]

Part IIB

Do not attempt more than two questions from Part IIB.

Question 15

Let G be a group and let g be an element of G.

(a) If H is a subset of G containing n elements, show that the set

$$g^{-1}Hg = \{g^{-1}hg : h \in H\}$$

also has n elements. [3]

(b) If H is a subgroup of G, show that $g^{-1}Hg$ is also a subgroup of G. [3]

(c) Let X be the set of all n-element subgroups of G. Show that if $H \in X$, $g \wedge H = g^{-1}Hg$ defines an action of G on X, and hence a homomorphism ϕ from G to the group of permutations of the set X. [4]

(d) (i) Use part (c) to construct a homomorphism ϕ from the dihedral group D_4 to the group S_5. [2]

(ii) Write down a non-identity element of D_4 belonging to the kernel of ϕ, giving a brief explanation. [3]

Question 16

(a) Find all Abelian groups of order 2 450 (up to isomorphism), expressing each of them in *both* canonical *and* p-primary form. [7]

(b) Write down the groups that you found in part (a) which possess a subgroup isomorphic to $\mathbb{Z}_{50}$, justifying your answer. [4]

(c) For each of the groups that you found in part (a), state how many elements of order 35 it possesses. (You need *not* give a justification.) [4]

Question 17

Let G be a group of order 117.

(a) By considering the Sylow subgroups of G, or otherwise, show that G has a normal subgroup, K, of order 13. [3]

(b) What possibilities are there for the number of Sylow 3-subgroups of G ? [3]

(c) By applying the Correspondence Theorem to a suitable quotient group of G, show that G has a normal subgroup of order 39. [5]

(d) Assuming that G has a normal subgroup of order 9, find the only possibilities for the group G, expressing each as a product of cyclic groups. [4]

[END OF QUESTION PAPER]

Please complete the identification grid below.

Examination No.								
Personal Identifier								

M336/K

Figure Sheet

Answer Sheet for Questions 11 and 13

Question 11

Please read carefully the instructions given for Question 11 of the question paper

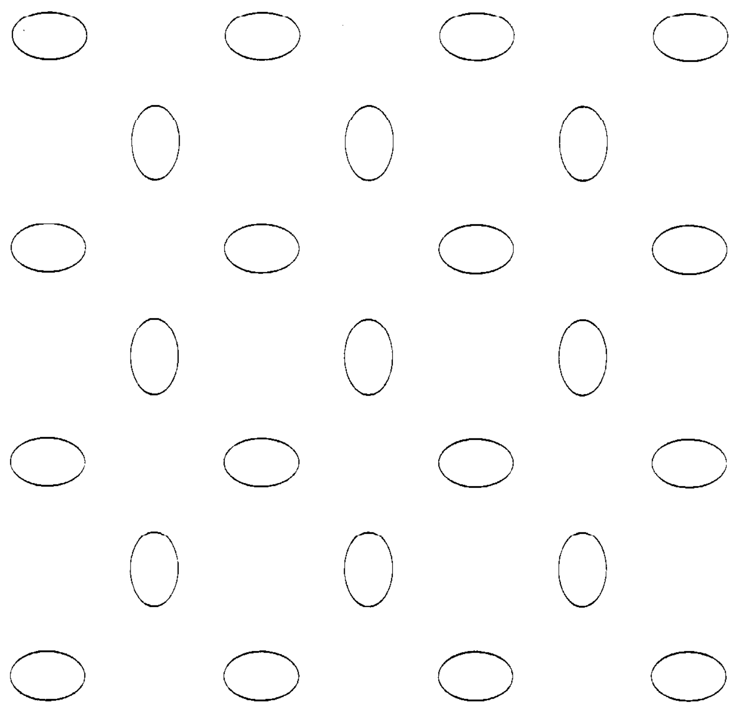


Figure 1

Question 13
Please read carefully the instructions given for Question 13 of the question paper

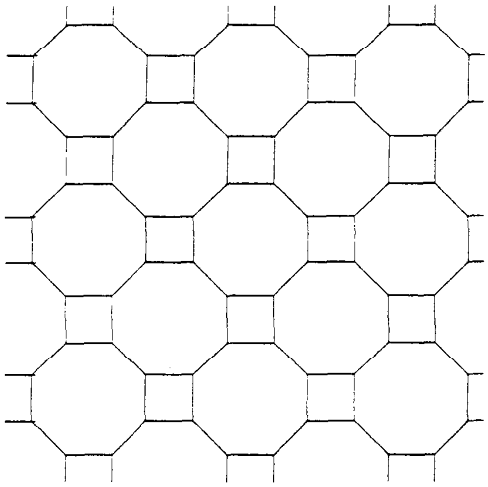


Figure 2

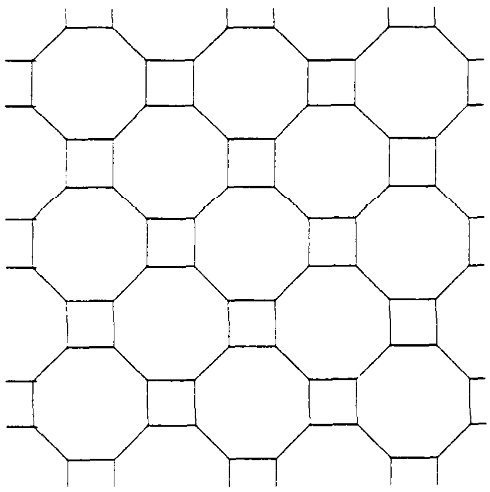


Figure 3

For examiner's use	