M336/G

The Open University

Third Level Course Examination 1996
Groups and Geometry

Thursday, 17th October, 1996 **10.00am–1.00pm**

Time allowed: 3 hours

There are TWO parts to this examination. You should attempt BOTH.

You may attempt ALL the questions in Part I. You should answer not more than THREE questions from Part II, including not more than TWO questions from Part IIA and not more than TWO questions from Part IIB.

You are advised to spend about 90 minutes on Part I and 80 minutes on Part II, and to leave yourself about 10 minutes for checking. Part I carries 55% of the total marks while Part II carries 45% of the total marks.

You are advised to show all your working and to give reasons for all your answers unless a question is explicitly phrased 'Write down...'. You should begin each answer on a new page of the answer book.

Note: a Figure Sheet is provided with this question paper for use in answering Questions 11 and 13.

At the end of the examination

Check that you have written your name, examination number and personal identifier on each answer book used and on any separate sheets used, *particularly the Figure Sheet*. **Failure to do so may mean that your paper cannot be identified.** Attach all your answer books and the Figure Sheet together, using the fastener provided.

Write the numbers of the questions that you have attempted in the spaces provided on the front of the answer book.

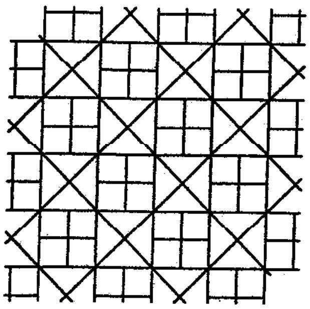
Part I

You may attempt* ALL *the questions in this part and are advised to spend about* 90 minutes *on it.

The marks for each question are given beside the question.

Question 1

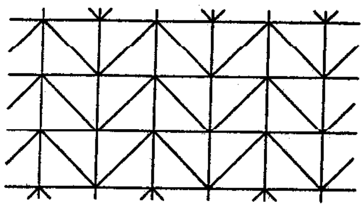
(a) Write down all the tile types and vertex types in the following tiling $\mathcal{S}$.



$\mathcal{S}$

[3]

(b) The following monohedral tiling, $\mathcal{T}$, is formed from $\mathcal{R}_4$ by bisecting each tile of $\mathcal{R}_4$ to form a pair of triangular tiles.



$\mathcal{T}$

(i) Write down the number of non-congruent tile shapes that are produced if $\mathcal{T}$ is subjected to the affine transformation

$$(x, y) \longmapsto (2x, y).$$ [1]

(ii) Write down the *maximum* number of non-congruent tile shapes that can be produced from $\mathcal{T}$ by an affine transformation. [1]

Question 2

The group G is given by the presentation

$$G = \langle r, s : r^8 = e, s^2 = r^4, sr = r^7 s (= r^{-1} s) \rangle$$

and the elements are written in standard form as

$$r^m s^n, \quad m = 0, 1, \ldots, 7, \quad n = 0, 1.$$

Note that G is *not* D_8.

(a) Express the product

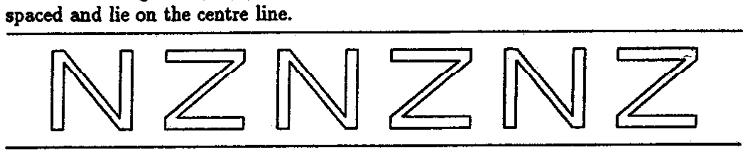
$$sr^3 s^{-1}$$

in standard form. [3]

(b) Find which of the elements of the form r^m commute with s. [2]

Question 3

(a) In the following frieze, $\mathcal{F}$, the centres of the 'N' and 'Z' motifs are equally spaced and lie on the centre line.



Use the algorithm of *Unit IB3* to determine the type of $\mathcal{F}$. [4]

(b) Write down the type of the frieze that would result from $\mathcal{F}$ if every 'Z' motif were moved the same small distance to the left so that the centres of the motifs are no longer equally spaced. (You need not give a justification for your answer.) [1]

Question 4

Let G be a group in which *every* non-identity element has order 2. Show that G is Abelian. [5]

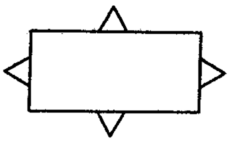
Question 5

Consider the following rectangle whose symmetry group G is of order 4.

(a) Write down the cycle index of G, considered as acting on the four corners A, B, C and D. [1]

(b) Write down the cycle index of G, considered as acting on the four sides AB, BC, CD and DA. [2]

(c) A rectangular piece of clear, colourless glass has four congruent, isosceles triangles of coloured glass stuck to it, one at the centre of each side.



Each triangle can be red (R) or green (G). Write down the pattern inventory for the configurations that can appear. [2]

Question 6

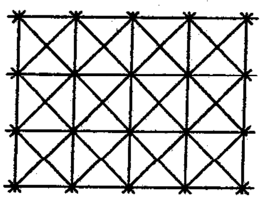
Write down all cyclic subgroups of order 4 of the direct product

$\mathbb{Z}_4 \times \mathbb{Z}_8$;

your answer should express each subgroup as $\langle (a, b) \rangle$ for a suitable element (a, b) of $\mathbb{Z}_4 \times \mathbb{Z}_8$. [5]

Question 7

Let $\mathcal{T}$ be the Laves tiling $[4, 8, 8]$:



$\mathcal{T}$

(a) Write down $n_v(\mathcal{T})$. [1]

(b) Using a theorem from *Unit GE2*, or otherwise, find $n_t(\mathcal{T})$. [2]

(c) Use the properties of the vertex-edge diagram of $\mathcal{T}$ to find $n_e(\mathcal{T})$. [2]

Question 8

The finitely presented Abelian group A is defined by

$$A = \langle a, b, c : 2a + 8b + 4c = 0, 4a + 6b + 8c = 0, 2a + 2b + 6c = 0 \rangle.$$

(a) Write down the integer matrix representing A. [1]

(b) Use the Reduction Algorithm to reduce the matrix to diagonal form. [3]

(c) Hence express A as a direct product of non-trivial cyclic groups in canonical form. [1]

Question 9

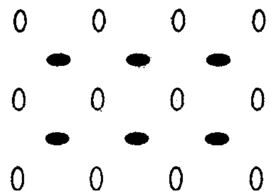
Let L be a plane lattice. ***Without*** assuming Theorem 5.3 of ***Unit GE3*** (the classification of plane lattices), show that if L has a rotation of order 3 about a vertex, then it has a rotation of order 6 about that vertex. [5]

Question 10

Let G be a ***non-Abelian*** group of order 50. Show that the centre $Z(G)$ of G has order at most 5. [5]

Question 11

Consider the wallpaper pattern W below, an enlarged copy of which is reproduced as Figure 1 on the Figure Sheet.



W

(a) On Figure 1 on the Figure Sheet, draw a basic rectangle of W and shade in a generating region. [3]

(b) Write down the wallpaper type of W. (You need not give a justification.) [2]

Part II

You may attempt not more than **THREE** *questions from this part, and you are advised to spend about* **80 minutes** *on it.*

You may choose not more than **TWO** *of your three questions from Part IIA and not more than* **TWO** *from Part IIB.*

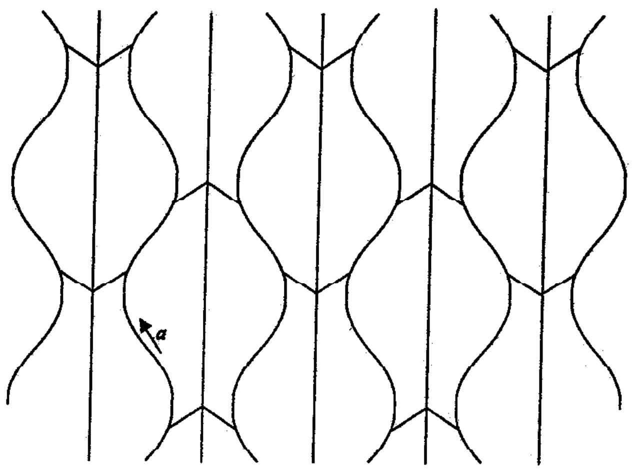
Each question carries 15% of the marks for the whole examination, and an indication of the allocation of marks within each question is given beside the question.

Part IIA

Do not attempt more than two questions from Part IIA.

Question 12

Consider the following transitive tiling $\mathcal{T}$.



$\mathcal{T}$

(a) Starting at the marked edge, trace round the inside of the tile, then around the outside, writing the appropriate edge side labels. Draw a copy of this tile in your answer book with the edge side labels clearly shown. [3]

(b) Hence write down the incidence symbol of $\mathcal{T}$. [3]

(c) How many edge side orbits are there under the action of $\Gamma(\mathcal{T})$? [1]

(d) How many edge orbits are there under the action of $\Gamma(\mathcal{T})$? Identify an edge orbit with stabilizer $\{e, h\}$ and one with stabilizer $\{e, r\}$. [3]

(e) *Neither* of the following can be the incidence symbol of a transitive tiling. For each one, explain briefly why not.

(i) $\overrightarrow{a}\,\overrightarrow{b}\,\overrightarrow{c}$

$\overleftarrow{b}\,\overleftarrow{c}\,\overleftarrow{a}$

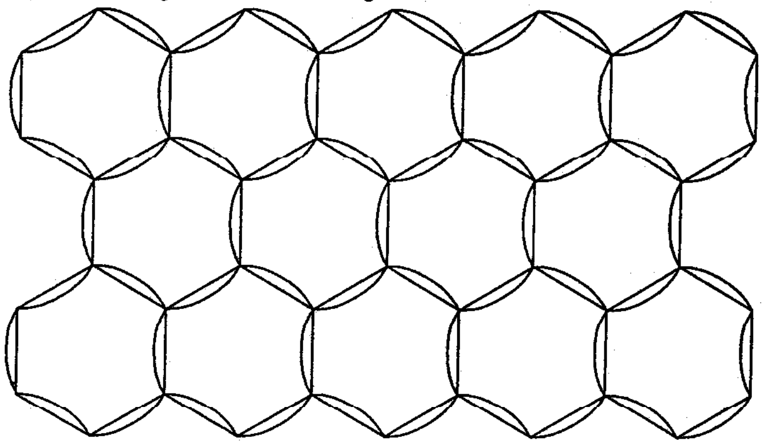
(ii) $\overrightarrow{a}\,\overrightarrow{a}\,b$.

$\overleftarrow{a}\,\overleftarrow{a}\,b$

[5]

Question 13

Consider the following wallpaper pattern W, which has been formed from the tiling $\mathcal{R}_6$ by changing each edge into a small segment of a circle. Two copies of W are reproduced as Figures 2 and 3 on the Figure Sheet.



W

(a) How many orbits of rotation centres are there under the action of the *translation* group $\Delta(W)$?
Clearly mark *one* representative of each orbit, using Figure 2 on the Figure Sheet. [3]

(b) How many orbits of rotation centres are there under the action of the *symmetry* group $\Gamma(W)$? [1]

(c) Carry out the algorithm described in the audio tape for *Unit GE4* to determine the type of W. You should explain at each stage your answer to the question posed by the algorithm. [5]

(d) By shading some of the segments of circles and leaving others unshaded, it is possible to alter the wallpaper type of the pattern, to produce a type of wallpaper pattern with no non-trivial rotation symmetry but with indirect symmetries.
EITHER turn Figure 3 on the Figure Sheet into such a pattern
OR describe *very clearly* which segments should be shaded to achieve this result. [4]

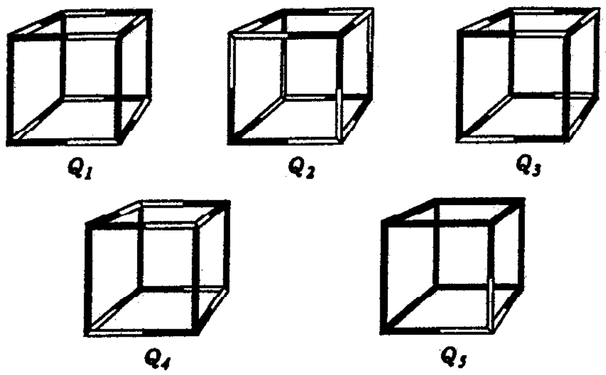
(e) Write down the wallpaper type of the pattern produced in part (d). [1]

(f) Suppose that, instead of shading any of the segments, you were to modify W by replacing every segment with a double segment:

Write down the wallpaper type of the pattern so produced. [1]

Question 14

Cubes Q_1 to Q_5 below are made with identical rods (of circular cross-section). Each rod is then painted so that *either* it is entirely black *or* it is black for exactly half its length and white for the other half.



(a) For each Q_i, write down $\Gamma^+(Q_i)$ as a standard group. [9]

(b) State which (if any) of the Q_i have *no* indirect symmetries. [2]

(c) State which (if any) of the Q_i have *central inversion* as an indirect symmetry. [2]

(d) For each i such that $\Gamma^+(Q_i)$ is cyclic, write down $\Gamma(Q_i)$ as a standard group or as the direct product of two standard groups. [2]

Part IIB

Do not attempt more than two questions from Part IIB.

Question 15

Let G be a group.

(a) Show that, for all $a, b, c, d \in G$,

$$abcd = a(bcb^{-1})bd.$$ [1]

Let H be a *normal* subgroup of G and let K be a subgroup of G.

(b) Show that the set

$$HK = \{hk : h \in H, k \in K\}$$

is a subgroup of G [7]

(c) For each of the following conditions, give an example of a finite group G, with a non-trivial, proper normal subgroup H and a non-trivial, proper subgroup K which also satisfy:

(i) $G = HK$ and $H \cap K = \{e\}$, [3]

(ii) $G = HK$, $H \cap K \neq \{e\}$ and G is *non-Abelian*. [4]

Question 16

(a) Find all Abelian groups of order 1000 (up to isomorphism), expressing each of them *both* in canonical *and* in p-primary form. [9]

(b) Which of the groups that you found in part (a) possess a subgroup isomorphic to

$$\mathbb{Z}_5 \times \mathbb{Z}_5?$$

Explain your answer carefully. [6]

Question 17

Let G be a *non-Abelian* group of order $4p$, where p is a prime number greater than 3.

(a) By considering the Sylow subgroups of G, or otherwise, show that G has a normal subgroup, K, of order p and find the number of Sylow 2-subgroups of G.
Hint: you may find Theorem 4.1 of *Unit GR5* useful. [8]

(b) By applying the Correspondence Theorem to a suitable quotient group of G, show that G has a normal subgroup of order $2p$. [4]

Now, let G act by conjugation on the set

$$\{H_1, H_2, \ldots\}$$

of its Sylow 2-subgroups.

(c) Show that the stabilizer of H_i is H_i, for each i. [3]

[END OF QUESTION PAPER]