M336/C

The Open University

Third Level Course Examination 1994
Groups and Geometry

Tuesday, 18th October, 1994 **10.00am–1.00pm**

Time allowed: 3 hours

There are **TWO** parts to this examination. You should attempt **BOTH**.

You may attempt **ALL** the questions in Part I. You should answer not more than **THREE** questions from Part II, including not more than **TWO** questions from Part IIA and not more than **TWO** questions from Part IIB.

You are advised to spend about 90 minutes on Part I and 80 minutes on Part II, and to leave yourself about 10 minutes for checking. Part I carries 55% of the total marks while Part II carries 45% of the total marks.

You are advised to show all your working and to give reasons for all your answers. Begin each answer on a new page of the answer book.

Note: a Figure Sheet is provided with this question paper for use in answering Questions 11 and 13.

At the end of the examination

Check that you have written your name, examination number and personal identifier on each answer book usedand on any separate sheets used. **Failure to do so may mean that your paper cannot be identified.** Attach all your answer books and the Figure Sheet together, using the fastener provided.

Write the numbers of the questions that you have attempted in the spaces provided on the front of the answer book.

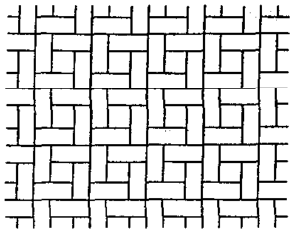
Part I

You may attempt ALL *the questions in this part and are advised to spend about* **90 minutes** *on it.*

The marks for each question are given beside the question.

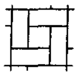
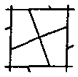
Question 1

(a) Write down all the tile types and vertex types in the following tiling $\mathcal{T}$.



$\mathcal{T}$ [3]

(b) Let $\mathcal{T}'$ be the tiling obtained from $\mathcal{T}$ by shrinking each square tile to a point. That is, each large square consisting of five tiles of $\mathcal{T}$ is altered

from  to 

State whether $\mathcal{T}'$ is

(i) vertex-uniform, [1]

(ii) tile-uniform. [1]

Question 2

The group G is given by the presentation

$$G = \langle r, s : r^4 = e, s^2 = r^2, sr = r^3 s \rangle$$

and the elements are written in standard form as

$$r^m \text{ and } r^m s, \quad m = 0, \ldots, 3.$$

Note that G is *not* D_4.

(a) Express the product

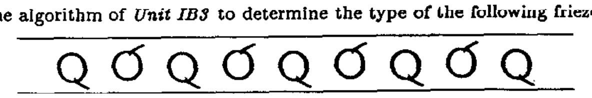
$$(r^2 s)\,(rs)$$

in standard form. [3]

(b) Express $(r^3 s)^{-1}$ in standard form. [2]

Question 3

Use the algorithm of *Unit IB3* to determine the type of the following frieze.



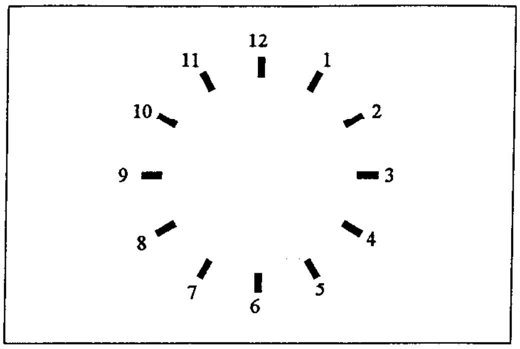
Your answer should make clear which questions you ask and what the answers are. [5]

Question 4

Let G be a group and let H and K be subgroups of G. Prove that $H \times K$ is a subgroup of the group $G \times G$. [5]

Question 5

In the following diagram, twelve 'hour marks' are evenly spaced round a circle whose centre coincides with the centre of the surrounding rectangle.



(a) Regarding the hour numbers as position labels for the hour marks, write down the orbits of the hour marks under the action of the group of *direct* symmetries of the surrounding rectangle. [2]

(b) Under the same conditions as for part (a), write down the orbits of the hour marks under the action of the group of *all* symmetries of the rectangle. [1]

(c) If the surrounding shape were a square, rather than a rectangle, write down the orbits of the hour marks under the action of the group of *all* symmetries of the square. [2]

Question 6

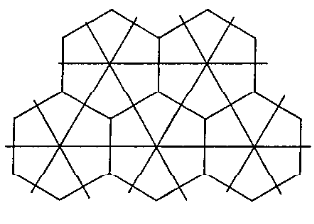
Explain why the group given by the direct product

$$\mathbb{Z}_{10} \times \mathbb{Z}_{15}$$

has only one subgroup of order six. [5]

Question 7

Let $\mathcal{T}$ be the following Laves tiling, $[3, 4, 6, 4]$.



(a) Write down $n_t(\mathcal{T})$. [1]

(b) Using a theorem from *Unit GE2*, or otherwise, find $n_v(\mathcal{T})$. [2]

(c) Use the properties of the tile-edge diagram of $\mathcal{T}$ to find $n_e(\mathcal{T})$. [2]

Question 8

A finitely presented Abelian group A is represented by the integer matrix

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

(a) Write down the group A as a direct product of cyclic groups and hence as a direct product of cyclic groups in canonical form. [3]

(b) Write down the 2-primary and 3-primary components of A. [2]

Question 9

Let L be the hexagonal lattice $L(\mathbf{a}, \mathbf{b})$, where

$$\mathbf{a} = (1, 0), \quad \mathbf{b} = (1/2, \sqrt{3}/2).$$

Let

$$\mathbf{c} = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}, \quad \mathbf{d} = \frac{2}{3}\mathbf{a} + \frac{2}{3}\mathbf{b},$$

so that $\mathbf{c}$ and $\mathbf{d}$ are 3-centres.
Write down, using any convenient notation:

(a) a reflection in $\Gamma(L)$ which maps $\mathbf{c}$ to $\mathbf{d}$; [1]

(b) a rotation of order 2 in $\Gamma(L)$ which maps $\mathbf{c}$ to $\mathbf{d}$; [1]

(c) a rotation of order 6 in $\Gamma(L)$ which maps $\mathbf{c}$ to $\mathbf{d}$; [1]

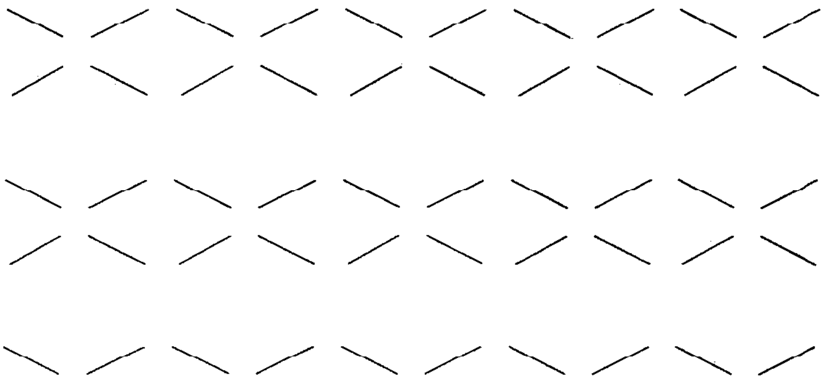
(d) a glide reflection in $\Gamma(L)$ which maps $\mathbf{c}$ to $\mathbf{d}$. [2]

Question 10

Classify all Abelian groups of order 36 by writing each as a direct product of cyclic groups in canonical form. [5]

Question 11

Figure 1 on the Figure Sheet is a copy of the figure below which depicts a wallpaper pattern.



On Figure 1 on the Figure Sheet:

(a) draw a basic rectangle of the wallpaper pattern; *[1]*

(b) mark, with a clearly visible dot or cross, ***exactly one*** representative of each orbit of 2-centres; *[2]*

(c) mark ***exactly one*** representative of each orbit of reflection axes (i.e. draw a line along an axis from each orbit). *[2]*

Part II

You may attempt not more than **THREE** *questions from this part, and you are advised to spend about* **80 minutes** *on it.*

You may choose not more than **TWO** *of your three questions from Part IIA and not more than* **TWO** *from Part IIB.*

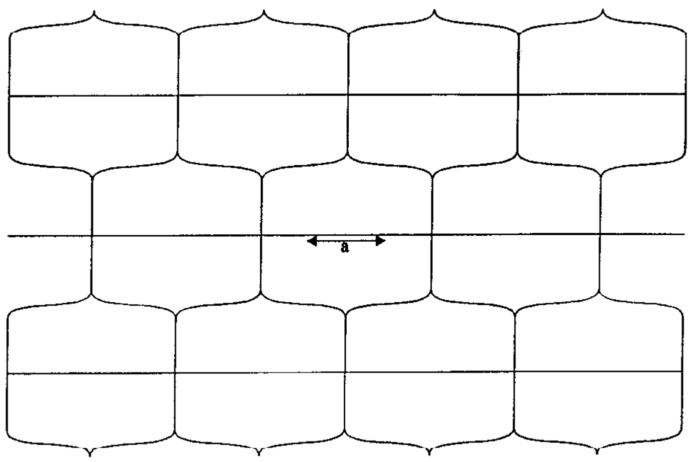
Each question carries 15% of the marks for the whole examination and an indication of the allocation of marks within each question is given beside the question.

Part IIA

Do not attempt more than two questions from Part IIA.

Question 12

Consider the following transitive tiling, $\mathcal{T}$.



(a) Starting at the marked edge, trace around the inside of the tile, then around the outside, writing the appropriate edge side labels. Draw a copy of this tile in your answer book, with the labels clearly shown. *[3]*

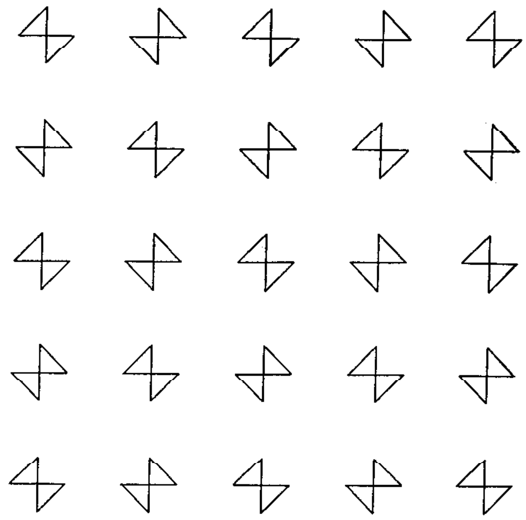
(b) Hence write down the incidence symbol of $\mathcal{T}$. *[3]*

(c) Find the tile stabilizer of $\mathcal{T}$. *[3]*

(d) Find the stabilizer of each edge orbit. *[6]*

Question 13

Consider the following wallpaper pattern, W.



(a) Carry out the algorithm described in the audio tape for *Unit GE4* to determine the type of the above pattern. You should explain carefully, at each step, ***both*** your answers to the questions posed by the algorithm *and* the reasons for your answers. [5]

(b) Using Figure 2 of the Figure Sheet (which is a copy of W), draw a basic parallelogram for W.
Also on Figure 2 of the Figure Sheet, shade in a generating region for the pattern which lies within the basic parallelogram that you have drawn. [5]

(c) Also on Figure 2 on the Figure Sheet, indicate *one* representative of each orbit of 4-centres and 2-centres. (Label them with the labels '4' and '2' respectively.)
Draw one glide axis in each direction for which such axes exist. [5]

Question 14

Find the type of each of the following Bravais lattices.

(a) $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$ where

$$\mathbf{a} = (2,0,0), \quad \mathbf{b} = (0,0,1), \quad \mathbf{c} = (0,1,0).$$ [3]

(b) $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$ where

$$\mathbf{a} = (1,0,0), \quad \mathbf{b} = (0,3,0), \quad \mathbf{c} = (1/2, 3/2, 2).$$ [4]

(c) $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$ where

$$\mathbf{a} = (2,1,0), \quad \mathbf{b} = (2,-1,0), \quad \mathbf{c} = (0,1,3).$$ [4]

(d) $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$ where

$$\mathbf{a} = (2,0,0), \quad \mathbf{b} = (1,\sqrt{3},0), \quad \mathbf{c} = (1,\sqrt{3}/3, 3/2).$$ [4]

Part IIB
Do not attempt more than two questions from Part IIB.

Question 15

(a) Let G be a group and let H be a subgroup of G.

(i) Prove that

$$H \cap Z(G) \subseteq Z(H),$$

where $Z(G)$ is the centre of G and $Z(H)$ is the centre of H. **[3]**

(ii) Show, by giving a suitable counterexample, that it is *not* necessarily true that

$$Z(H) \subseteq H \cap Z(G).$$

Explain your counterexample carefully. **[4]**

(b) For each of the following assertions, state whether it is true or false and give a proof or counterexample, as appropriate.

(i) If G is a group and H is a non-trivial proper subgroup of G, then the centre of G is contained in the centre of H, that is

$$Z(G) \subseteq Z(H).$$ [4]

(ii) If G is a group and H_1 and H_2 are subgroups of G, then

$$Z(H_1) \cap Z(H_2) \subseteq Z(H_1 \cap H_2).$$ [4]

Question 16

(a) Find all Abelian groups (up to isomorphism) of order 700, expressing each of them *both* in canonical *and* in p-primary form. [8]

(b) For each of the groups found in part (a), state how many isomorphism classes of subgroups of order 100 it possesses. Give full reasons for your answers. [7]

Question 17

Let G be a group of order 30.

(a) Use Sylow's theorems to show that G *either* has a normal subgroup of order 3 *or* a normal subgroup of order 5. [8]

(b) Apply the Correspondence Theorem to appropriate quotient groups of G to show that G must have a normal subgroup of order 15. [7]

[END OF QUESTION PAPER]