MT365/E

The Open University

Third-level Course Examination 1999
Graphs, Networks and Design

Wednesday 13th October 1999 10.00 am – 1.00 pm

Time allowed: 3 hours

There are TWO parts to this paper.

52% of the available marks are assigned to Part 1 (4 marks per question) and 48% are assigned to Part 2 (12 marks per question). You should not expect to be awarded a distinction unless you obtain high marks on both Part 1 and Part 2.

In Part 1 you should attempt as many questions as you can. Please begin each new question on a new page, and indicate clearly the number of the question you are attempting.

In Part 2 you should attempt not more than FOUR questions, including at least one question from each section. Please begin each new question on a new page, and *write the numbers of the Part 2 questions you attempt on the front page of the answer book for Part 2*.

Write your answers to Parts 1 and 2 in separate answer books. Additional answer books are available from the invigilator, if needed.

At the end of the examination

Attach together, using the paper fastener provided, the answer books in which you have answered questions from Part 1 and Part 2.

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

YOU MUST NOT USE A CALCULATOR IN THIS EXAMINATION.

Part 1

Part 1 carries 52% of the total marks for the examination (4 marks per question). Answer as many questions as you can from this part. It will help the examiners if you answer the questions in the order in which they are set.

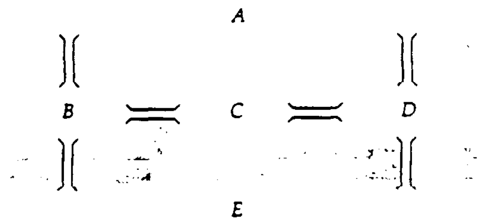
Write your answers in one of the answer books provided.

DO NOT use the same answer book for Part 1 as for Part 2.

Please begin each Part 1 question on a new page.

Question 1

A town consists of five land areas linked by six bridges, as shown:



(a) Draw a graph G that represents this arrangement of land areas and bridges, and write down its degree sequence.

(b) Does there exist a walk around the town that crosses each bridge exactly once? (Give a reason for your answer.)

(c) Draw another graph, not isomorphic to G, with the same degree sequence as G.

Question 2

Consider the following matrices:

$$\mathbf{A}_1 = \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0&0&0&1&1&1&1 \\ 2 & 0&0&0&1&1&1&1 \\ 3 & 0&0&0&1&1&1&1 \\ 4 & 1&1&1&0&0&0&0 \\ 5 & 1&1&1&0&0&0&0 \\ 6 & 1&1&1&0&0&0&0 \\ 7 & 1&1&1&0&0&0&0 \end{array} \qquad \mathbf{A}_2 = \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0&1&1&0&0&0&0 \\ 2 & 1&0&1&0&0&0&0 \\ 3 & 1&1&0&0&0&0&0 \\ 4 & 0&0&0&0&1&1&1 \\ 5 & 0&0&0&1&0&1&1 \\ 6 & 0&0&0&1&1&0&1 \\ 7 & 0&0&0&1&1&1&0 \end{array}$$

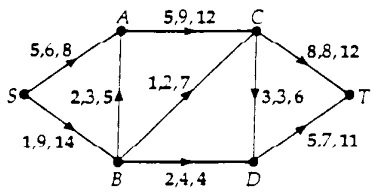
Describe or draw:

(a) a graph with adjacency matrix $\mathbf{A}_1$;

(b) a graph with adjacency matrix $\mathbf{A}_2$.

Question 3

The following diagram shows a feasible flow in a network with lower and upper capacities:



(a) Find a maximum flow in this network, stating the flow-augmenting paths used (they may be found by inspection).

(b) Write down a minimum cut and its capacity.

Question 4

Write down the answers to the following questions.

(a) Which regular tiling is dual to the equilateral triangle tiling?

(b) How many 4-ominoes are there?

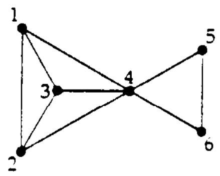
(c) Which regular polyhedron is also an antiprism?

(d) Which regular polyhedron is dual to the tetrahedron?

Question 5

(a) Write down the number of distinct labelled spanning trees in
(i) K_3; (b) K_4.

(b) Consider the following labelled graph G:



How many distinct labelled spanning trees does G have?
(Give a brief reason for your answer.)

Question 6

A project consists of eight activities A–H with the following durations (in hours):

activity	A	B	C	D	E	F	G	H
duration	12	5	7	6	8	10	9	2

It is required to find the minimum number of workers needed to complete the project in 20 hours.

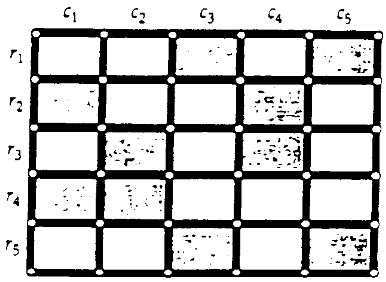
(a) Find the solution given by the first-fit algorithm.

(b) Find the solution given by the first-fit decreasing algorithm.

(In each part, draw a diagram to show how workers are assigned to projects.)

Question 7

Consider the following braced rectangular framework:

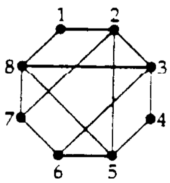


(a) Draw the corresponding bipartite graph and deduce that the framework is not rigid.

(b) One further bay needs to be braced to achieve rigidity. Find *two* possible choices for this bay.

Question 8

Consider the following graph G:



(a) Determine whether G is planar.

(b) Write down the chromatic number $\chi(G)$.

Question 9

Three factories A_1, A_2, A_3, can supply 12, 14, 16 units of product, respectively. The demands for these products at four warehouses, b_1, b_2, b_3, b_4 are 10, 13, 9, 10 units, respectively. The costs of transporting one unit of product from each factory to each warehouse are given in the following cost matrix:

	b_1	b_2	b_3	b_4
A_1	26	24	18	13
A_2	9	5	4	10
A_3	15	17	13	9

Find the first revised cost matrix and construct the first partial graph.

Do not proceed with the algorithm.

Question 10

Three of the eight codewords of a linear code C are:

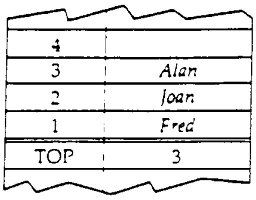
00110101, 10011100, 11100100.

(a) Find the other five codewords of C.

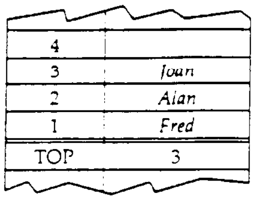
(b) Determine the number of errors that the code C can correct.

Question 11

Diagram (A) shows a computer store containing a stack *s*:



(A)



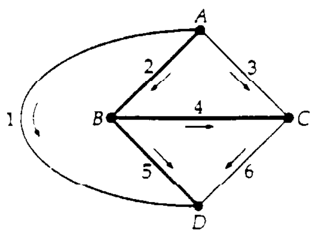
(B)

(a) Write down the result of applying the operation TOP(POP(PUSH(*Bridget,s*))) to the stack *s*.

(b) Write down the operation (of the data type stack) that, when applied to the stack *s*, produces the stack in diagram (B) above.

Question 12

The following diagram shows an oriented graph with a spanning tree indicated by thick lines:



Given that the fundamental *cycle* equations are

$$v_1 - v_5 - v_2 = 0, \quad v_3 - v_4 - v_2 = 0, \quad v_4 + v_6 - v_5 = 0,$$

draw another oriented graph, and indicate a spanning tree, for which the fundamental *cutset* equations are

$$i_1 - i_5 - i_2 = 0, \quad i_3 - i_4 - i_2 = 0, \quad i_4 + i_6 - i_5 = 0.$$

Question 13

Consider the following three incomplete block designs:

1	2	3	4		1	2	3	4		1	2	3	4
A	*A*	*D*	*C*		*A*	*C*	*A*	*B*		*A*	*B*	*A*	*F*
B	*B*	*E*	*E*		*B*	*D*	*E*	*D*		*D*	*F*	*C*	*C*
C	*D*	*F*	*F*		*C*	*E*	*F*	*F*		*E*	*D*	*E*	*B*
	design 1					design 2					design 3		

Write down the answers to the following questions.

(a) Which designs are connected?

(b) Which designs are (0, 1)-designs?

(c) Which designs are resolvable?

(d) Which two designs are isomorphic?

Part 2

Part 2 is divided into three sections. You should attempt **not more than FOUR** *questions from this part, including* **at least one from each section.**

Each question in this part is allotted 12 **marks.**

Show all your working.

To help the examiners, please write the numbers of the questions you have attempted in Part 2 at the foot of the front cover of your answer book for this part.

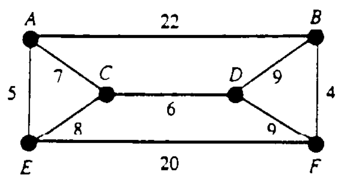
DO NOT use the same answer book as you used for Part 1.

Please begin each Part 2 question on a new page.

Section A Graphs

Question 14

(a) Consider the following weighted graph G.



(i) Use Kruskal's algorithm to find a minimum-weight spanning tree T for G. State clearly the order in which the edges are chosen, and why. *(4 marks)*

(ii) If Prim's algorithm is used to find a minimum-weight spanning tree T, and the initial vertex put into T is chosen at random, how likely is it that the shortest edge of G is the *last* edge to be put into the tree? Explain your answer. *(5 marks)*

(b) Let H be a connected, weighted graph with 50 vertices and 51 edges. Briefly describe a method of finding a minimum-weight spanning tree for H that is much more efficient than either Prim's or Kruskal's algorithm for this particular case. *(3 marks)*

Question 15

Let K_n be the complete graph on n vertices, labelled $1, 2, \ldots, n$.

(a) Explain why $\chi'(K_n) \geq n - 1$, for any $n > 1$. *(1 mark)*

(b) Suppose that each edge ij of K_n (where $n > 1$) is given the 'colour'

$$\begin{cases} i + j & \text{if } i + j \leq n \\ i + j - n & \text{if } i + j > n \end{cases}$$

Show that at each vertex the incident edges have distinct colours, and deduce that $\chi'(K_n) \leq n$. *(4 marks)*

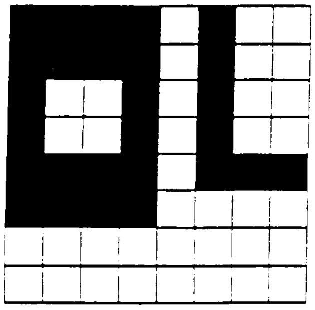
(c) Write down the maximum number of edges *of the same colour* that can be present in *any* edge-colouring of K_n, when n is even and when n is odd. *(1 mark)*

(d) Hence show that, if n is odd, then $\chi'(K_n) = n$. *(2 marks)*

(e) Show that, if n is odd and K_n has an n-edge-colouring, then a different colour must be *missing* at each vertex; deduce that, if n is odd, then $\chi'(K_{n+1}) = n$. *(4 marks)*

Question 16

Consider the following image displayed on a 4-screen.



(a) Draw the pruned quad tree that stores this image. *(4 marks)*

(b) On your quad tree for part (a), mark the path arising from the steps of the north neighbour algorithm when determining the north neighbour of the pixel in row 5, column 6. *(4 marks)*

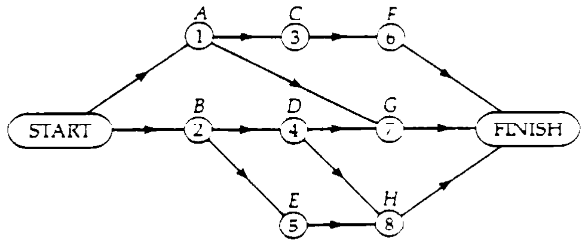
(c) At one point in applying the algorithm of part (b), the current vertex is the root vertex. Write down the labels in the 'path' stack at this point, and underline (or ring) the label at the TOP of the stack. *(1 mark)*

(d) A computer consultant is testing an implementation of the north neighbour algorithm on a 5-screen, and notices that at one point the 'path' stack contains the labels *ABC*<u>*D*</u> (with *D* at the TOP). Explain why this shows that the implementation must be faulty. *(3 marks)*

Section B Networks

Question 17

Consider the following activity network in which activities are represented by vertices:



(a) Write down a list of precedence relations for the activities of this network; your list should contain no unnecessary information. *(2 marks)*

(b) Write down the activities that must be completed before activity H can be started. *(1 mark)*

The durations (in days) of the activities are as follows:

activity	A	B	C	D	E	F	G	H
duration	4	2	3	6	8	7	2	10

(c) Use the critical path construction algorithm to find the critical path and the minimum completion time of the project. *(6 marks)*

(d) Find the float of activity D. *(1 mark)*

(e) If, because of unforeseen circumstances, activity D takes 13 days to complete instead of 6 days, find the delay in the completion of the project. *(2 marks)*

Question 18

Four applicants apply for four jobs. The jobs for which each applicant is qualified are shown in the following table.

applicants	jobs
A_1	B_2, B_3
A_2	B_1, B_2, B_4
A_3	B_1
A_4	B_1, B_4

(a) Use the marriage theorem to prove that it is possible to assign all four applicants to jobs for which they are qualified. *(4 marks)*

(b) Use the maximum matching algorithm to find a maximum matching of applicants to jobs for which they are qualified, starting with the matching $A_1\ B_2$, $A_2\ B_1$. Show all your working. *(6 marks)*

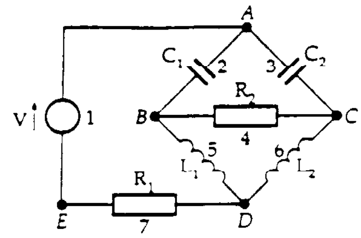
(c) Now suppose that there is an additional applicant A_5, and that the ability of each applicant to do each job for which he/she is qualified is represented by a 'cost'. These costs are shown in the following matrix; a lower cost indicates a higher ability.

	B_1	B_2	B_3	B_4
A_1	–	2	1	–
A_2	2	4	–	4
A_3	3	–	–	–
A_4	3	–	–	4
A_5	–	2	1	3

Explain briefly how the problem can be modified so that the Hungarian algorithm for the assignment problem can be used to find an allocation with minimum total cost. *(2 marks)*

Question 19

(a) Consider the following network:



(i) Draw a fully-labelled oriented graph that represents this network. *(2 marks)*

(ii) Write down the component equations for components 1, 2 and 7. *(2 marks)*

(iii) Using the spanning tree with edges corresponding to components 1, 2, 3 and 7, write down *two* fundamental cycle equations that include the voltage across component 2. *(2 marks)*

(iv) Using the spanning tree in part (iii), write down *two* fundamental cutset equations that include the current flowing through component 4. *(2 marks)*

(b) The oriented graph representing **another** electrical network has the following incidence matrix:

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{cccccc} AB & AE & BC & BD & CD & DE \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

(i) Taking vertex C as the reference vertex, and using the spanning tree with edges AB, AE, CD and DE, find the reduced incidence matrix and partition it in the form

$$\mathbf{B}_0 = [\mathbf{B}_t \mid \mathbf{B}_c],$$

where $\mathbf{B}_t$ has columns corresponding to branches and $\mathbf{B}_c$ has columns corresponding to chords. *(2 marks)*

(ii) Find the fundamental cutset matrix, given that the inverse of $\mathbf{B}_t$ is

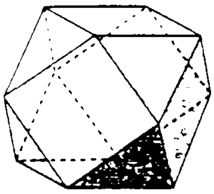
$$\mathbf{B}_t^{-1} = \begin{array}{c} \\ AB \\ AE \\ CD \\ DE \end{array} \begin{array}{c} \begin{array}{cccc} A & B & D & E \end{array} \\ \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 \end{bmatrix} \end{array}.$$

(2 marks)

Section C Geometry

Question 20

The following diagram illustrates a cuboctahedron:



(a) Write down the numbers of vertices, edges, triangular faces and square faces of a cuboctahedron. *(3 marks)*

(b) Sketch a plane drawing of the graph of the cuboctahedron, and verify that Euler's formula holds for your drawing. *(6 marks)*

(c) Explain why all the faces of the dual of a cuboctahedron are quadrilaterals. Write down the degrees of the vertices of the dual, and the number of vertices of each degree. *(3 marks)*

Question 21

Consider a planar kinematic system with n links and j joints, where:

no link has multiplicity greater than s;

for $0 \le r \le s$, there are exactly n_r r–ary links;

all joints are binary joints.

For such a system, it can be shown that:

$$n_0 + n_1 + n_2 + \cdots + n_r + \cdots + n_s = n \qquad (1)$$

and

$$n_1 + 2n_2 + 3n_3 + \cdots + rn_r + \cdots + sn_s = 2j \qquad (2)$$

(a) Consider the possible planar kinematic systems with 8 links, 10 revolute pairs and mobility 1, whose interchange graphs are 2-connected (that is, they remain connected after the removal of any vertex). For such a system, $s \le n/2$.

Write down the particular forms of equations (1) and (2) for such systems. *(4 marks)*

(b) There are sixteen distinct systems of the type in part (a):

nine systems, each with 4 binary links and 4 ternary links;

five systems, each with 5 binary links, 2 ternary links and 1 quaternary link;

two systems, each with 6 binary links and 2 quaternary links.

For each of these three classes, select one representative system, sketch its interchange graph, and hence sketch the system. *(6 marks)*

(c) Choose one of your three interchange graphs from part (b) and draw a spanning tree T. Determine the mobility of the planar kinematic system represented by T (considered as an interchange graph). *(2 marks)*

Question 22

(a) Find all the codewords of the code C with parity check matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(3 marks)

(b) Find all the codewords of C^*, the dual of C. *(2 marks)*

(c) Are the codes C and C^* equivalent?

(Justify your answer briefly.) *(2 marks)*

(d) Find a generator matrix of a systematic code C' equivalent to C. *(2 marks)*

(e) A codeword of C is transmitted and the binary word

101001

is received. Assuming that at most two errors have occurred, determine the codewords of C that may have been sent. *(3 marks)*

[END OF QUESTION PAPER]