MT365/ T

Third-level Course Examination 1998
Graphs, Networks and Design

Thursday 22nd October 1998 10.00 am – 1.00 pm

Time allowed: 3 hours

There are TWO parts to this paper.

52% of the available marks are assigned to Part 1 (4 marks per question) and 48% are assigned to Part 2 (12 marks per question). You should not expect to be awarded a distinction unless you obtain high marks on both Part 1 and Part 2.

In Part 1 you should attempt as many questions as you can. Please begin each new question on a new page, and indicate clearly the number of the question you are attempting.

In Part 2 you should attempt not more than FOUR questions, including at least one question from each section. Please begin each new question on a new page, and *write the numbers of the Part 2 questions you attempt on the front page of the answer book for Part 2.*

Write your answers to Parts 1 and 2 in separate answer books. Additional answer books are available from the invigilator, if needed.

At the end of the examination

Attach together, using the paper fastener provided, the answer books in which you have answered questions from Part 1 and Part 2.

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

YOU MUST NOT USE A CALCULATOR IN THIS EXAMINATION.

Part 1

Part 1 carries 52% of the total marks for the examination (4 marks per question). Answer as many questions as you can from this part. It will help the examiners if you answer the questions in the order in which they are set.

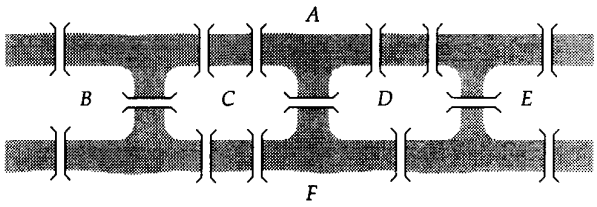
Write your answers in one of the answer books provided.

DO NOT use the same answer book for Part 1 as for Part 2.

Please begin each Part 1 question on a new page.

Question 1

The city of Combinatoria consists of six land areas linked by fourteen bridges, as follows:



The citizens of Combinatoria wish to go for a walk that crosses each bridge exactly once.

(a) Explain why such a walk is not possible.

(b) Show how the building of an extra bridge enables such a walk to be made, and write it down. Can such a walk begin and end at the same point? (Justify your answer.)

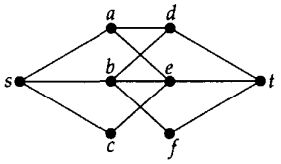
Question 2

Let G be a k-regular simple graph with n vertices.

Write down an expression for the number of edges of G when $k = 3$ and when $k = 4$, and sketch such a graph G for each of the two smallest possible values of n in each case.

Question 3

Consider the following graph G:



(a) Write down $\kappa(G)$ and $\lambda(G)$.

(b) Use an appropriate form of Menger's theorem to find the maximum number of vertex-disjoint st-paths in G.

Question 4

For each of the following, draw five points A, B, C, D, E whose convex hull $\langle A, B, C, D, E \rangle$ is

(a) a line-segment;

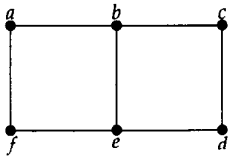
(b) a triangle;

(c) a quadrilateral;

(d) a tetrahedron.

Question 5

(a) Consider the following labelled graph G.



(1) How many labelled spanning trees does G have?

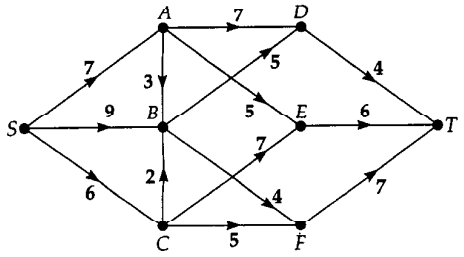
(2) How many non-isomorphic spanning trees does G have?

(b) How many labelled spanning trees does the complete graph K_6 have?

(Give brief reasons for your answers.)

Question 6

Consider the following weighted digraph D.



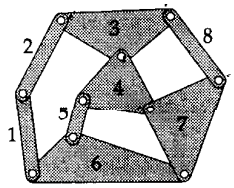
The shortest path algorithm is applied to the digraph D to find a shortest path from S to T.

(a) Write down the potentials assigned to each vertex in the order in which they are assigned. (No explanation is required.)

(b) Write down a shortest path from S to T.

Question 7

Consider the following kinematic system:

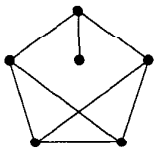


(a) Draw the interchange graph of the system.

(b) Calculate the mobility of the system when it is considered as a planar kinematic system containing only revolute pairs.

Question 8

Consider the following graph G.



Write down (no explanation is required):

(a) the number of faces in any plane drawing of G;

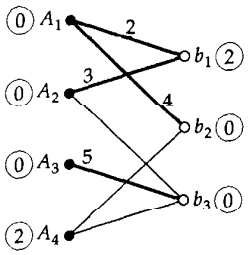
(b) the chromatic number of G;

(c) the chromatic index of G;

(d) the independence number of G.

Question 9

The Hungarian algorithm for the transportation problem is applied to an example with four supply vertices A_1, A_2, A_3, A_4 and three demand vertices b_1, b_2, b_3. At a certain stage in the application, the following partial graph has been obtained, with a flow denoted by thick lines; the circled numbers are the current supplies and demands.



Carry out the labelling procedure (which leads to breakthrough), and the flow-augmenting procedure, to obtain the required flow pattern.

(Present your answer with the aid of two clearly labelled diagrams.)

Question 10

Consider the following code C:

{0000000000, 0110110010, 1001001101, 1111111111}.

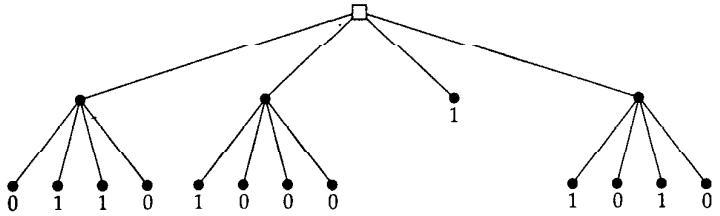
(a) How many errors can be detected and corrected by the code C? (Justify your answer.)

(b) A codeword is transmitted and the binary word 0101001101 is received. Determine the codeword that is most likely to have been transmitted.

(c) Is C a linear code? (Justify your answer.)

Question 11

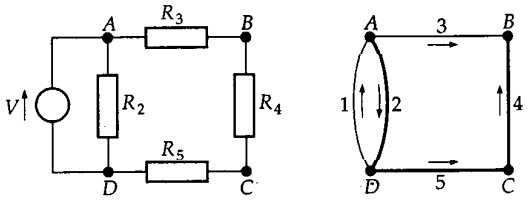
(a) Draw the 4 × 4 screen image that corresponds to the following pruned quad tree.



(b) Draw the pruned quad tree that stores the screen image obtained by rotating the image in part (a) through two right angles.

Question 12

Consider the following electrical network and oriented graph:



(a) Write down the incidence matrix of the oriented graph.

(b) Using the spanning tree with edges 2, 4 and 5 and the vertex C as reference vertex, write down the reduced incidence matrix partitioned into two parts, the first corresponding to the branches and the second to the chords.

Question 13

Let Δ be the incomplete block design

1	2	3	4	5	6	7	8	9	10	11	12
A	A	C	B	G	B	A	A	C	D	B	D
C	E	E	C	H	F	F	B	D	F	E	E
I	H	F	G	I	H	G	D	H	I	I	G

(a) Is Δ a resolvable design?

If so, write down the replicates; if not, explain why not.

(b) Is Δ a balanced design?

If so, write down the value of λ; if not, explain why not.

Part 2

Part 2 is divided into three sections. You should attempt **not more than FOUR** *questions from this part, including* **at least one from each section.**

Each question in this part is allotted **12 marks.**

Show all your working.

To help the examiners, please write the numbers of the questions you have attempted in Part 2 at the foot of the front cover of your answer book for this part.

DO NOT use the same answer book as you used for Part 1.

Please begin each Part 2 question on a new page.

Section A Graphs

Question 14

The distances (in miles) between six towns are as follows.

	A	*B*	*C*	*D*	*E*	*F*
A	–	23	33	25	8	11
B	23	–	9	6	23	31
C	33	9	–	3	18	23
D	25	6	3	–	13	21
E	8	23	18	13	–	4
F	11	31	23	21	4	–

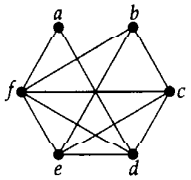
(a) Use Prim's algorithm, starting with town *A*, to find a minimum connector joining these towns. (Your answer should state clearly the order in which the edges are chosen, and why.) *(5 marks)*

(b) Consider the travelling salesman problem for these six towns. Use the heuristic algorithm for travelling salesman problems, starting with town *A*, to obtain an *upper bound* for the solution. *(4 marks)*

(c) Use Kruskal's algorithm to find a *lower bound* for the solution to the travelling salesman problem for these six towns. *(3 marks)*

Question 15

Consider the following graph G:



(a) By using the cycle method for testing planarity, show that G is planar. Give a plane drawing of G and verify Euler's formula for your drawing. *(4 marks)*

(b) The greedy colouring algorithm for vertex colouring is being used to colour the vertices of G. The colours are 1, 2, 3, ... State the colour assigned to each vertex, and write down the resulting upper bound for the chromatic number $\chi(G)$. *(2 marks)*

(c) Find the correct value of $\chi(G)$, giving reasons for your answer, and compare your value with the bound given by Brooks' theorem. *(3 marks)*

(d) Find the chromatic index $\chi'(G)$, and compare your answer with the bounds given by Vizing's theorem. *(3 marks)*

Question 16

(a) A university wishes to recruit new students. It has been decided to spend £10,000 on advertisements and must decide in which of four newspapers A, B, C, D to advertise. The finance department has ascertained the cost of each advertisement, and the market research department has estimated the number of students that are likely to be recruited by advertising in each newspaper. The results are as follows.

newspaper	A	B	C	D
cost (£1000)	2	5	7	3
number of recruits	200	900	1400	400

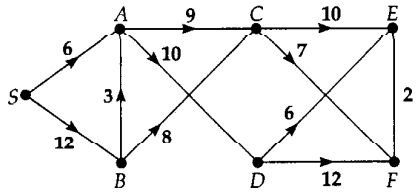
Use the branch-and-bound method to advise the university where advertisements should be placed so as to maximize the estimated number of recruits. *(9 marks)*

(b) The Handbook describes the branch-and-bound method as: 'Break the problem down into a set of subproblems and determine the bound for each'. In the first branching in your solution to part (a), describe these subproblems and state the appropriate bound. Is it an upper or a lower bound? *(3 marks)*

Section B Networks

Question 17

Consider the following network in which the number next to each arc or edge indicates the capacity of that arc or edge.



(a) Explain how the network can be converted to a basic network, assuming that we are concerned with the value of the flow from S to F. *(1 mark)*

(b) By finding flow-augmenting paths, find a maximum flow from S to F and state its value. (Your answer should include a list of all the flow-augmenting paths you use, together with the values of the flows you send along them.) Write down a corresponding minimum cut. *(8 marks)*

(c) Now suppose that we are also interested in finding a maximum flow from S to E. Use the max-flow min-cut theorem to show that the value of a maximum flow from S to E is the same as the value of a maximum flow from S to F. *(3 marks)*

Question 18

(a) The durations (in days) of the activities of a project are:

activity	A	B	C	D	E	F	G	H
duration	5	6	4	10	2	9	3	1

The precedence relations are

D must follow E

E must follow B and C

F must follow A and D

G must follow E

H must follow G.

(1) Use the algorithms given in *Networks 2* to number the vertices and to draw a *fully-labelled* activity network in which the activities are represented by vertices. *(5 marks)*

(2) Find a critical path by inspection, or otherwise. Write down the length of the critical path. *(1 mark)*

(b) An activity network represents a project with 8 activities $A, B, \ldots, H$. There are exactly two critical paths in the network, one via the vertices B, C, E and H, and the other via the vertices B, D, F and H. The floats of activities A and G are 6 days and 10 days, respectively. There is no restriction on the number of workers available. For each of the following questions, show your working or give a brief justification for your answer.

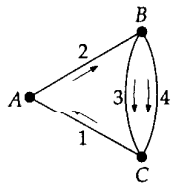
(1) What is the value of the float of activity C? *(2 marks)*

(2) It is required to reduce the completion time of the project by reducing the duration of just one activity. Which of the activities could be used? *(2 marks)*

(3) By how many days would the completion time of the project be increased if the duration of activity A were increased by 10 days? *(2 marks)*

Question 19

The following oriented graph represents an electrical network which contains a voltage source and three resistors with resistances R_2, R_3 and R_4.



(a) The edge AC represents a voltage source that produces a voltage V. Draw a fully labelled circuit diagram of the electrical network. (It is not necessary to show currents and voltages for the resistors.) *(2 marks)*

(b) Using the spanning tree with edges 1 and 4:

(1) write down the fundamental cycle equations and the corresponding fundamental cycle matrix. (The matrix should be written with rows and columns labelled and with columns corresponding to branches placed first.)

(2) write down the fundamental cutset equations and the corresponding fundamental cutset matrix. (The columns of the matrix should be in the same order as in the fundamental cycle matrix.) *(6 marks)*

(c) What additional equations are required in order to determine all the currents and voltages? Write down these equations. *(2 marks)*

(d) Suppose that two additional resistors are added to the electrical network so that the oriented graph has two corresponding edges BD and CD added, where D is a new vertex. Show that this new oriented graph has exactly 12 different spanning trees. (It is not sufficient simply to list all the spanning trees.) *(2 marks)*

Section C Design

Question 20

A robot design team decides to explore a range of new designs for robots to be used on a laboratory bench. The designs are to be restricted to planar kinematic systems in which the only joints are binary joints, all of which are revolute pairs.

(a) The team decides that the mobility of the system should be in the range 3 to 5, inclusive. Using the appropriate mobility criterion, tabulate the possible numbers of links and joints that can occur together in such a system with mobility 3, and describe in one sentence the general relationship between the number of links and the number of joints. *(3 marks)*

(b) The team eventually decides to have a system with mobility 5 whose interchange graph is a tree structure.

(1) Determine the numbers of links and joints in such a tree structure.

(2) Sketch all possible tree structures with mobility 5. Treating these as interchange graphs, sketch the corresponding planar kinematic systems. *(9 marks)*

Question 21

(a) Find the codewords of the code C with generator matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$ *(2 marks)*

(b) Is the code C cyclic? Is the code C systematic? (Justify your answers briefly.) *(3 marks)*

(c) Write down a parity check matrix for C. *(1 mark)*

(d) Let C' be the code obtained from C using the $[\mathbf{a} \mid \mathbf{a} + \mathbf{b}]$ construction, taking $\mathbf{a}$ in C and $\mathbf{b}$ in the dual code C^*. Find a parity check matrix for C'. *(4 marks)*

(e) A codeword is transmitted and the binary word 1101000100 is received. Determine the codeword of C' that is most likely to have been sent. *(2 marks)*

Question 22

(a) Find two mutually orthogonal 3×3 Latin squares. *(2 marks)*

(b) Show how these Latin squares can be used to find a 4-replicate $(0, 1)$-design Δ with 9 varieties. *(4 marks)*

(c) Write down the concurrence matrix $\mathbf{C}$ of Δ, and determine whether Δ is balanced. *(2 marks)*

(d) A coffee house wishes to try out nine new blends of coffee. Four regular customers are chosen to test the blends; each is to compare three blends at a time. Show how the design Δ can be used to ensure that each customer tastes every blend and that each blend is compared with every other blend by exactly one customer. *(4 marks)*

[END OF QUESTION PAPER]