MT365/J

Third-level Course Examination 1997
Graphs, Networks and Design

Friday 17th October 1997 10.00 am – 1.00 pm

Time allowed: 3 hours

There are TWO parts to this paper.

52% of the available marks are assigned to Part 1 (4 marks per question) and 48% are assigned to Part 2 (12 marks per question). You should not expect to be awarded a distinction unless you obtain high marks on both Part 1 and Part 2.

In Part 1 you should attempt as many questions as you can. Please begin each new question on a new page, and indicate clearly the number of the question you are attempting.

In Part 2 you should attempt not more than FOUR questions, including at least one question from each section. Please begin each new question on a new page, and *write the numbers of the Part 2 questions you attempt on the front page of the answer book for Part 2.*

Write your answers to Parts 1 and 2 in separate answer books. Additional answer books are available from the invigilator, if needed.

At the end of the examination

Attach together, using the paper fastener provided, the answer books in which you have answered questions from Part 1 and Part 2.

Check that you have written your name, personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

YOU MUST NOT USE A CALCULATOR IN THIS EXAMINATION.

Part 1

Part 1 carries 52% of the total marks for the examination (4 marks per question). Answer as many questions as you can from this part. It will help the examiners if you answer the questions in the order in which they are set.

Write your answers in one of the answer books provided.

DO NOT use the same answer book for Part 1 as for Part 2.

Please begin each Part 1 question on a new page.

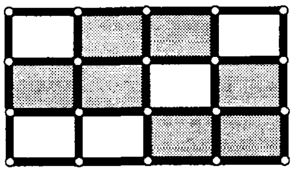
Question 1

Each of the following situations can be represented by a bipartite graph; draw an appropriate graph in each case:

(a) a job assignment problem in which four applicants apply for five jobs:

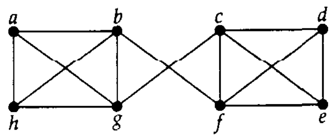
applicant	job
1	*a, c*
2	*b, c, e*
3	*d, e*
4	*a, b, d*

(b) a braced rectangular framework:



Question 2

Consider the following graph *G*:



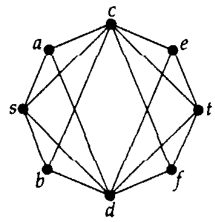
(a) Verify the handshaking lemma for *G*.

(b) Is *G* Eulerian? If so, write down an Eulerian trail; if not, explain why not.

(c) Is *G* Hamiltonian? If so, write down a Hamiltonian cycle; if not, explain why not.

Question 3

Consider the following graph G:



(a) Write down the values of $\kappa(G)$ and $\lambda(G)$, and justify each answer by stating an appropriate set of vertices/edges whose removal disconnects G.

(b) Write down (without explanation)

(1) two vertex-disjoint st-paths;

(2) four edge-disjoint st-paths.

Question 4

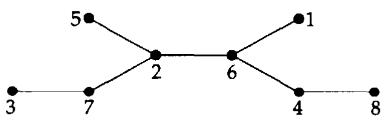
(a) A polyhedron has t triangular faces, s square faces and h hexagonal faces, and exactly four faces meet at each vertex. Write down expressions for the total numbers of vertices (v), edges (e) and faces (f) in terms of t, s and h.

(b) By applying Euler's polyhedron formula, prove that

$$t = 2h + 8.$$

Question 5

Consider the following tree T:



(a) Classify the tree T as *central* or *bicentral*.
(Explain how you obtain your answer, and show which vertex/vertices form the centre/bicentre.)

(b) Write down the Prüfer sequence corresponding to T.

Question 6

A project consists of eight activities A–H with the following durations (in days). There are no precedence relations.

activity	A	B	C	D	E	F	G	H
duration	9	4	7	2	10	5	8	3

It is required to find the minimum number of workers needed to complete this project in 12 days. Each activity is to be completed by a single worker.

(a) What answer is given to this problem by the first-fit packing method?

(b) What answer is given by the first-fit decreasing packing method?

(In each case, draw a diagram to show how tasks are allocated to workers.)

Question 7

A ladder is constrained so that its two feet are always in point contact with the ground and its upper end always rests against a vertical wall at two points of contact. There is no friction at the points of contact, and the ladder is able to move so that the contacts with the wall and the ground are maintained.

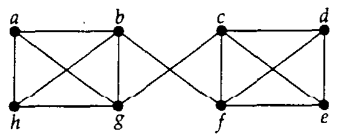
(a) Write down the number of freedoms of the ladder.

(b) List a set of parameters describing the position of the ladder.

(c) Write down the number of further constraints that are needed in order to immobilize the ladder.

Question 8

Consider the following graph G:



(a) Construct a plane drawing of G, and verify that Euler's formula holds for your drawing.

(b) Find the chromatic number of G, giving reasons for your answer.

Question 9

Four applicants *a*, *b*, *c*, *d* are rated for their ability to carry out four tasks *w*, *x*, *y*, *z*, according to the following cost matrix (with lower cost implying greater ability).

	w	*x*	*y*	*z*
a	9	5	8	4
b	9	2	8	4
c	4	2	9	2
d	5	9	3	4

Construct the first revised cost matrix and the first partial graph used in applying the Hungarian algorithm to find the optimum assignment of applicants to jobs. (Do NOT proceed with the algorithm.)

Question 10

Consider the following codes *A* and *B*:

$A = \{0000, 0011, 1100, 1111\}$
$B = \{0000, 1111\}$

Write down (no explanation is required):

(a) a generator matrix for code *A*;

(b) a parity check matrix for code *B*;

(c) the codewords obtained from codes *A* and *B* by the [**a** | **a** + **b**] construction, where **a** is a codeword of code *A*, and **b** is a codeword of code *B*.

Question 11

The following diagram shows a computer store containing a stack *s*.

4	
3	*graphs*
2	*networks*
1	*design*
TOP	3

(a) Write down the results of applying each of the following basic operations of the data type stack to the stack *s*:

(1) TOP(POP(POP(*s*)));

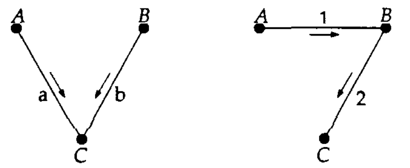
(2) TOP(POP(PUSH(MT365, *s*)));

(3) TOP(PUSH(MT365, POP(*s*))).

(b) Write down the value of DEPTH(PUSH(MT365, POP(*s*))).

Question 12

A 3-terminal component in an electrical network can be represented by either of the following oriented graphs:



(a) Find the voltages v_1 and v_2 of the second graph in terms of the voltages v_a and v_b of the first graph.

(b) Find the currents i_1 and i_2 of the second graph in terms of the currents i_a and i_b of the first graph.

Question 13

Are there any balanced block designs with the following parameters?

(a) $v = 7, b = 7, r = 3, k = 3, \lambda = 1$;

(b) $v = 8, b = 10, r = 5, k = 4, \lambda = 2$.

In each case justify your answer *either* by writing down a balanced design with the given parameters, *or* by explaining why such a balanced design cannot exist.

Part 2

Part 2 is divided into three sections. You should attempt **not more than FOUR** *questions from this part, including* **at least one from each section.**

Each question in this part is allotted **12 marks.**

Show all your working.

To help the examiners, please write the numbers of the questions you have attempted in Part 2 at the foot of the front cover of your answer book for this part.

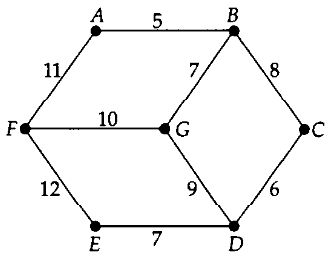
DO NOT use the same answer book as you used for Part 1.

Please begin each Part 2 question on a new page.

Section A Graphs

Question 14

(a) Consider the following weighted graph G:



(1) Find a minimum-weight spanning tree T for G by using Kruskal's algorithm. State clearly the order in which the edges are chosen, and why. *(4 marks)*

(2) Find a spanning tree T' for G by using the following 'cutting-down' algorithm:

Successively delete edges of maximum weight from G, subject to the condition that the graph remains connected.

State clearly the order in which the edges are deleted. Verify that $T' = T$. *(4 marks)*

(b) (1) Let H be *any* weighted graph, let S be a minimum-weight spanning tree of H, let e be any edge of H *not* in S, and let C be the cycle created by adding e to S. Show that e is an edge of maximum weight in C.

(2) Hence show that the 'cutting down' algorithm of part (a)(2) does indeed yield a minimum spanning tree. *(4 marks)*

Question 15

A **(5,3)-graph** is a connected graph which is regular of degree 3 and in which the shortest cycle has length 5.

(a) Name or draw one example of each of the following:

(1) a non-planar (5,3)-graph with 10 vertices;

(2) a planar (5,3)-graph with 20 vertices. (4 *marks*)

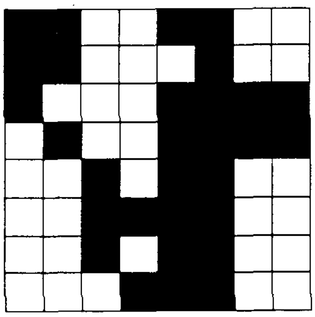
(b) Let G be a planar (5,3)-graph with n vertices.

(1) Use Euler's formula to prove that $n \geq 20$.

(2) Prove, by mathematical induction on the number of vertices, that the chromatic number of G satisfies $\chi(G) \leq 4$. (8 *marks*)

Question 16

Consider the following image displayed on a 4-screen:



(a) Draw the pruned quad tree that stores this image. (4 *marks*)

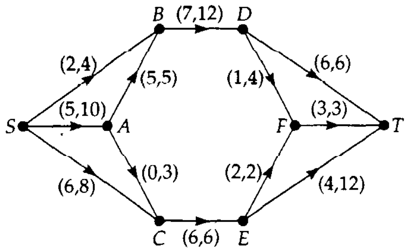
(b) Draw the quad tree that represents the original image reflected in the diagonal from top right to bottom left, by interchanging appropriate vertices of the quad tree obtained in part (a). (4 *marks*)

(c) On your quad tree for part (a), mark the path arising from the steps of the north neighbour algorithm when determining the north neighbour of the pixel in row 5, column 3. (4 *marks*)

Section B Networks

Question 17

Consider the following network in which the numbers next to an arc are the value of the flow through the arc and the capacity of the arc.



(a) Write down the value of the flow from S to T. (1 *mark*)

(b) Find, by inspection, a minimum cut in the network; hence, by using an appropriate theorem, determine the value of a maximum flow from S to T. (2 *marks*)

(c) Starting with the given flow, use the maximum flow algorithm to find a maximum flow from S to T. (Show all your working. It is not necessary to consider the vertices in alphabetical order.) (5 *marks*)

(d) It is required to increase the value of the maximum flow by increasing the capacity of just one arc. Show that there is only one arc which can be used to do this, and identify this arc. What is the largest value of the maximum flow which can be achieved in this way? (3 *marks*)

(e) Write down the lowest value of the maximum flow from S to T which can be obtained by decreasing the capacity of a single arc, and identify this arc. (1 *mark*)

Question 18

Five applicants apply for five jobs, but not every applicant is qualified to do every job. The following table lists the jobs which each of the applicants is qualified to do.

applicants	jobs
A_1	B_2, B_3, B_4
A_2	B_1, B_3
A_3	B_1, B_3, B_5
A_4	B_5
A_5	B_3, B_5

(a) Use an algorithm given in *Networks* 3 to find one possible maximum allocation of applicants to jobs for which they are qualified, starting with the matching $(A_1\,B_3, A_3\,B_1, A_5\,B_5)$. Show all your working and indicate clearly how the procedure of the algorithm terminates. (6 *marks*)

(b) Use the marriage theorem to show that

(1) any three applicants can be allocated to suitable jobs;

(2) there is just one set of four applicants that *cannot* be allocated to suitable jobs. (4 *marks*)

(c) In part (a) you were asked to use the algorithm to find one possible maximum allocation of applicants to jobs. Explain in two or three sentences how the algorithm can be used to find *all* of the possible maximum allocations. (2 *marks*)

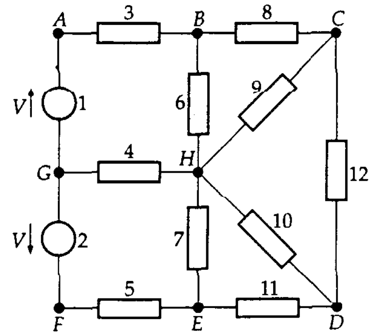
Question 19

(a) An electrical network consists of p 3-terminal components and q 2-terminal components connected together. The oriented graph of the network has n vertices.

(1) How many edges are there in the oriented graph?

(2) How many linearly independent voltage equations are there for the network? (3 *marks*)

(b) The following network consists of two identical voltage sources connected to ten identical resistors.



(1) Draw a fully-labelled oriented graph for the network and indicate a spanning tree. (3 *marks*)

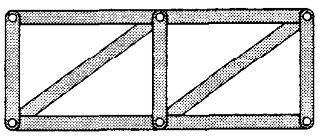
(2) Let C be the cutset whose edges correspond to components 8, 9, 10 and 11. By applying Kirchhoff's current law to appropriate *vertices*, show that the algebraic sum of the currents in C is zero. (3 *marks*)

(3) Suppose that the current flowing in component 6 has value $3x$ and the current flowing in component 8 has value $4x$. By applying Kirchhoff's current law to an appropriate *cutset* and using the symmetry of the network, find the value of the current flowing in component 4. (3 *marks*)

Section C Design

Question 20

Consider the following braced rectangular framework:



(a) By treating the framework as a planar kinematic system with only revolute joints, sketch its direct graph. (1 *mark*)

(b) Determine the number of possible new systems that can be obtained from the framework by expanding each ternary and each quaternary joint to produce a kinematic system in which all joints are binary. (4 *marks*)

(c) Sketch one example of the possible expanded systems described in part (b), and draw its interchange graph.

For your example, write down the numbers of links and joints of each type, and determine the mobility of the system. (7 *marks*)

Question 21

Consider the linear code C with the following generator matrix:

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

(a) Write down the length and rate of the code C. (2 *marks*)

(b) Is the code C systematic?
(Justify your answer briefly.) (2 *marks*)

(c) Find a parity check matrix for C. (4 *marks*)

(d) Write down the dimension of the dual code C^*. (1 *mark*)

(e) A codeword of C^* is transmitted and the binary word 11011001 is received. Find the error syndrome of the received word and hence determine the codeword of C^* that is most likely to have been transmitted. (3 *marks*)

Question 22

Consider the following balanced design Δ:

1	2	3	4	5	6	7	8	9	10	11	12
1	4	7	1	2	3	1	2	3	1	2	3
2	5	8	4	5	6	5	6	4	6	4	5
3	6	9	7	8	9	9	7	8	8	9	7

(a) Verify that $\lambda(v-1) = r(k-1)$ for Δ. (2 *marks*)

(b) Construct the dual Δ^* of Δ. (2 *marks*)

(c) Construct a 9×9 latin square by using the Steiner triple system construction with Δ. (4 *marks*)

(d) Each of nine drivers has to test drive nine cars $A, B, \ldots, I$. Show how the latin square constructed in part (c) can be used to timetable the drivers to test the cars in nine time slots. (4 *marks*)

[END OF QUESTION PAPER]