MT365/U

Third-level Course Examination 1995
Graphs, Networks and Design

Thursday 26th October 1995 2.30 pm – 5.30 pm

Time allowed: 3 hours

There are TWO parts to this paper.

52% of the available marks are assigned to Part 1 (4 marks per question) and 48% are assigned to Part 2 (12 marks per question). You should not expect to be awarded a distinction unless you obtain high marks on both Part 1 and Part 2.

In Part 1 you should attempt as many questions as you can. Please begin each new question on a new page, and indicate clearly the number of the question you are attempting.

In Part 2 you should attempt not more than FOUR questions, including at least one question from each section. Please begin each new question on a new page, and *write the numbers of the Part 2 questions you attempt on the front page of the answer book for Part 2.*

Write your answers to Parts 1 and 2 in separate answer books. Additional answer books are available from the invigilator, if needed.

At the end of the examination

Attach together, using the paper fastener provided, the answer books in which you have answered questions from Part 1 and Part 2.

Check that you have written your name, personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

YOU MUST NOT USE A CALCULATOR IN THIS EXAMINATION.

Part 1

Part 1 carries 52% of the total marks for the examination (4 marks per question). Answer as many questions as you can from this part. It will help the examiners if you answer the questions in the order in which they are set.

Write your answers in one of the answer books provided.

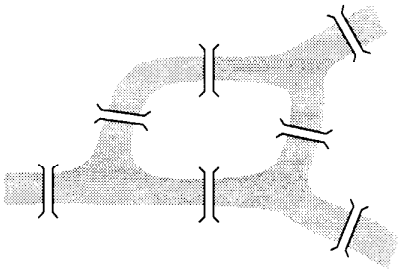
DO NOT use the same answer book for Part 1 as for Part 2.

Please begin each Part 1 question on a new page.

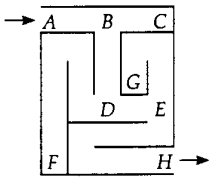
Question 1

Each of the following situations can be represented by a graph; draw an appropriate graph in each case:

(a) the arrangement of land areas and bridges:



(b) the possible routes that can be taken in tracing the following maze:



Question 2

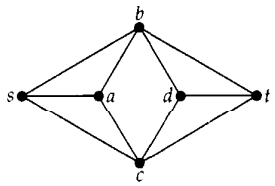
(a) Draw the graph G whose adjacency matrix is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

(b) Draw a *simple* graph with the same degree sequence as G.

Question 3

Consider the following graph G:



Write down (no explanation is required):

(a) three edge-disjoint st-paths;

(b) two vertex-disjoint st-paths;

(c) the value of $\lambda(G)$;

(d) the value of $\kappa(G)$.

Question 4

Write down the answers to the following questions.

(a) Which regular tiling is dual to the hexagonal tiling?

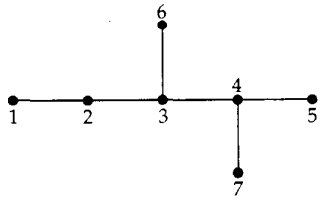
(b) Which regular polyhedron is dual to the cube?

(c) Which regular polyhedron is also a prism?

(d) Which regular polyhedron is also an anti-prism?

Question 5

Consider the following tree T:



(a) Classify the tree T as *central* or *bicentral*.
(You should explain how you obtain your answer, and show which vertex/vertices form the centre/bicentre.)

(b) Write down the Prüfer sequence corresponding to T.

Question 6

A project consists of eight activities A–H with the following durations (in days). There are no precedence relations.

activity	A	B	C	D	E	F	G	H
duration	8	3	9	1	2	6	7	4

It is required to find the minimum number of workers needed to complete this project in 10 days. Each activity is to be completed by a single worker.

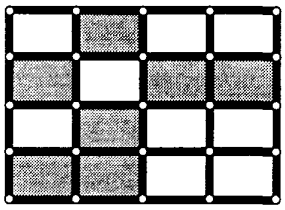
(a) What answer is given to this problem by the first-fit packing method?

(b) What answer is given by the first-fit decreasing packing method?

(In each part, draw a diagram to show how tasks are allocated to workers.)

Question 7

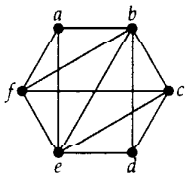
(a) Show that the following braced rectangular framework is rigid by constructing a suitable bipartite graph.



(b) Is the bracing shown above a minimum bracing?
(Give a reason for your answer.)

Question 8

Consider the following graph G:

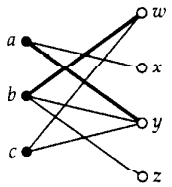


(a) Show that G is planar, by constructing a plane drawing of G.

(b) Find the chromatic number of G, giving reasons for your answer.

Question 9

In the following bipartite graph, the thick lines denote an initial matching:



(a) Write down *all* the alternating paths with respect to this matching.

(b) Write down the improved matchings obtained by using these alternating paths.

Question 10

Consider the following linear code:

$C = \{0000000, 0011101, 0101110, 0110011, 1001011, 1010110, 1100101, 1111000\}$.

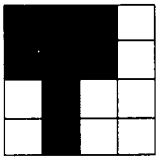
(a) Write down the minimum distance of this code.

(b) Write down the number of errors that this code can detect and correct.

(c) A codeword is transmitted and the binary word 0100101 is received. Write down the codeword that is most likely to have been transmitted.

Question 11

Consider the following image displayed on a screen:

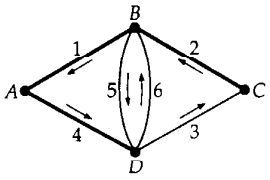


(a) Draw the quad tree that stores this image.

(b) Draw the quad tree that stores this image after it has been rotated anticlockwise through a right angle.

Question 12

The following oriented graph represents an electrical network with six components:



Using the spanning tree shown by thick lines, write down:

(a) any **two** fundamental cycles, and the corresponding voltage equations;

(b) any **two** fundamental cutsets, and the corresponding current equations.

Question 13

Are there any balanced block designs with the following parameters?

(a) $v = 6,\ b = 8,\ r = 4,\ k = 3,\ \lambda = 1$;

(b) $v = 7,\ b = 7,\ r = 3,\ k = 3,\ \lambda = 1$.

In each case, justify your answer *either* by writing down a balanced design with the given parameters, *or* by explaining why such a balanced design cannot exist.

Part 2

Part 2 is divided into three sections. You should attempt **not more than FOUR** *questions from this part, including* **at least one from each section.**

Each question in this part is allotted **12 marks.**

Show all your working.

To help the examiners, please write the numbers of the questions you have attempted in Part 2 at the foot of the front cover of your answer book for this part.

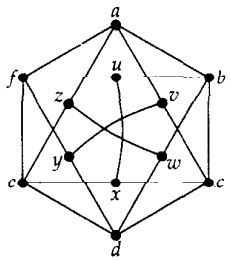
DO NOT use the same answer book as you used for Part 1.

Please begin each Part 2 question on a new page.

SECTION A GRAPHS

Question 14

Consider the following graph G:



(a) Explain why any Hamiltonian cycle of G must contain at least one of the edges ux, vy and wz. (4 *marks*)

(b) Write down a Hamiltonian cycle of G that includes the edge ux. (2 *marks*)

(c) Use the cycle method to determine whether G is a planar graph. (6 *marks*)

Question 15

The following table gives the distances (in kilometres) between five European cities.

	Geneva	Munich	Strasbourg	Venice	Zurich
Geneva	—	580	380	670	280
Munich	580	—	380	470	300
Strasbourg	380	380	—	800	230
Venice	670	470	800	—	570
Zurich	280	300	230	570	—

(a) Find a minimum connector for these cities, by using

(1) Kruskal's algorithm;

(2) Prim's algorithm, starting with Munich.

For each algorithm, state clearly the order in which the edges are chosen, and why. (6 *marks*)

(b) Use Kruskal's algorithm to find a lower bound for the solution to the travelling salesman problem for these cities.
(Explain how you obtain your answer.) (3 *marks*)

(c) In his travels, the travelling salesman needs to travel *directly* from Geneva to Munich. Describe briefly how the method of part (b) can be adapted so as to cater for this extra restriction.
(You are *not* required to calculate a new lower bound.) (3 *marks*)

Question 16

Let G be a plane drawing of a connected planar graph with n vertices, m edges and f faces, and suppose that the shortest cycle in G has length 6 or more.

(a) By counting the edges around each face, show that
$m \geq 3f$. (2 *marks*)

(b) By combining the results of part (a) with Euler's formula, prove that
$m \leq \frac{3}{2}(n-2)$. (3 *marks*)

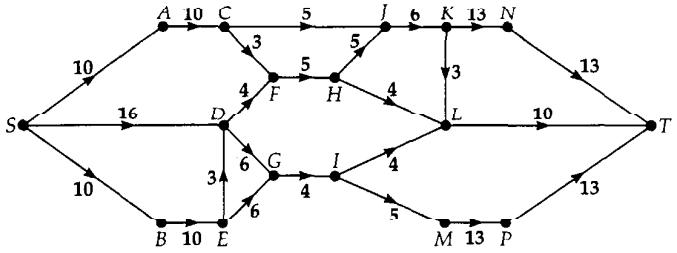
(c) By considering the average vertex degree in G, deduce from the result of part (b) that G has at least one vertex of degree 2 or less. (3 *marks*)

(d) Use the result of part (c) to prove that the chromatic number of G satisfies the inequality
$\chi(G) \leq 3$. (4 *marks*)

SECTION B NETWORKS

Question 17

Consider the following basic network in which the number next to each arc is the capacity of the arc:



(a) Draw a diagram showing a maximum flow from S to T, clearly indicating the flow along each arc. Explain carefully how you know that your flow is a maximum flow. (5 *marks*)

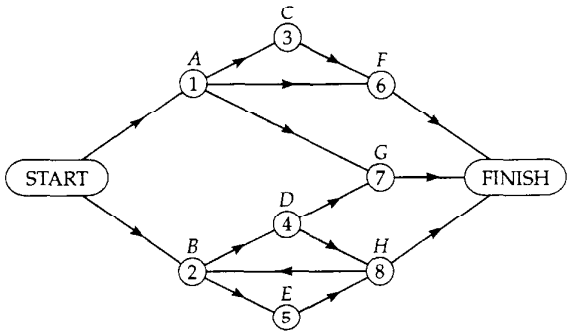
(b) For which arcs e is it true that *decreasing* the capacity of the single arc e by one unit (while leaving all other capacities unchanged) will decrease the value of the maximum flow from S to T?
(Explain your answer briefly.) (2 *marks*)

(c) For which arc e is it true that *increasing* the capacity of the single arc e by one unit (while leaving all other capacities unchanged) will increase the value of the maximum flow from S to T?
(Explain your answer briefly.) (2 *marks*)

(d) What is the largest value of the maximum flow that you can achieve by increasing the capacity of a single arc of the basic network?
(Explain your answer briefly.) (3 *marks*)

Question 18

Consider the following activity network:



(a) The activities of a project are represented by the vertices of the above activity network. However, the diagram has one incorrect arc and one unnecessary arc. Identify these, and explain why they are incorrect or unnecessary. (3 *marks*)

For the remaining parts of this question, these two arcs should be considered to have been removed. If the incorrect arc is not removed, you will not be able to answer the remaining parts of the question, but it is still possible to answer them with the unnecessary arc in place.

The durations (in days) of the activities of the project are as follows:

activity	A	B	C	D	E	F	G	H
duration	4	2	3	6	8	7	2	10

(b) Use the critical path construction algorithm to find the critical path and the minimum completion time of the project. (6 *marks*)

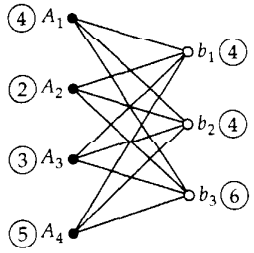
(c) Find the float of activity D. (1 *mark*)

(d) If, because of unforeseen circumstances, activity D takes 12 days to complete, instead of 6 days, find how long the whole project will be delayed. (2 *marks*)

Question 19

Four coal depots A_1, A_2, A_3, A_4 supply three steel works b_1, b_2, b_3 with coal, as modelled by the following bipartite graph; the circled numbers are the supplies and demands. The associated cost matrix gives the cost of supplying a unit quantity of coal from each depot to each steel works.

	④ b_1	④ b_2	⑥ b_3
④ A_1	5	7	5
② A_2	3	2	7
③ A_3	5	4	6
⑤ A_4	8	7	4



Use the Hungarian algorithm to determine how to supply the steel works from the coal depots at minimum cost, and find this cost.
(Show all your working.) (12 *marks*)

SECTION C DESIGN

Question 20

(a) Sketch all possible planar arrangements of regular polygons around a vertex, without gaps and in edge-to-edge contact, when only polygons with three, four or six sides are allowed. Write down the code for each such arrangement. (5 *marks*)

(b) Sketch all the distinct n-ominoes, for $1 \leq n \leq 4$. (3 *marks*)

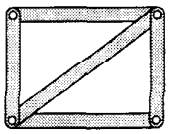
(c) If we represent all the 5-ominoes by graphs in which vertices represent squares and edges represent shared sides, sketch all the *non-isomorphic* graphs obtained. (4 *marks*)

Question 21

(a) (1) Draw a diagram showing a typical ternary joint formed with three binary links, clearly labelling all the links and joints.

(2) Sketch all possible expansions of this ternary joint into binary joints, labelling clearly the links and joints in each case. (4 *marks*)

(b) The following diagram shows a planar kinematic system.



(1) Write down the number and type of all links and joints in this system.

(2) Write down an appropriate mobility formula for this system, and hence derive its mobility.

(3) By expanding each non-binary joint into binary joints, derive two possible resulting systems. Sketch each system, together with its interchange graph.

(4) By treating the system as a braced rectangular framework, draw its bipartite graph and deduce whether the framework is rigid.
(Give a reason for your answer.) (8 *marks*)

Question 22

(a) Find all the codewords of the code C with the parity check matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(Explain how you obtain them.) (2 *marks*)

(b) Find all the codewords of C^*, the dual of C, and find a parity check matrix for C^*.
(Explain how you obtain them.) (4 *marks*)

(c) Construct a new code C' by using the $[\mathbf{a} \mid \mathbf{a}+\mathbf{b}]$ construction, taking $\mathbf{a}$ from C and $\mathbf{b}$ from C^*. Find a parity check matrix for C'.
(Explain how you obtain your answers.) (4 *marks*)

(d) State whether any of the codes C, C^* and C' is a cyclic code.
(No explanation is required.) (2 *marks*)

[END OF QUESTION PAPER]