The Open University

MT365/L

Third-level Course Examination 2000
Graphs, Networks and Design

Monday 16th October — 10.00 am – 1.00 pm

Time allowed: 3 hours

There are TWO parts to this paper.

52% of the available marks are assigned to Part 1 (4 marks per question) and 48% are assigned to Part 2 (12 marks per question). You should not expect to be awarded a distinction unless you obtain high marks on both Part 1 and Part 2.

Note: *In the wording of the questions,* 'write down' *or* 'state' *means 'write down without justification'; words such as* 'find', 'derive', 'explain' *or* 'calculate' *mean that we require you to show all your working.*

In Part 1 you should attempt as many questions as you can. Please begin each new question on a new page, and indicate clearly the number of the question you are attempting.

In Part 2 you should attempt not more than FOUR questions, including at least one question from each section. Please begin each new question on a new page, and *write the numbers of the Part 2 questions you attempt on the front page of the answer book for Part 2.*

Write your answers to Parts 1 and 2 in separate answer books. Additional answer books are available from the invigilator, if needed.

At the end of the examination

Attach together, using the paper fastener provided, the answer books in which you have answered questions from Part 1 and Part 2.

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

YOU MUST NOT USE A CALCULATOR IN THIS EXAMINATION.

Part 1

Part 1 carries 52% of the total marks for the examination (4 marks per question). Answer as many questions as you can from this part. It will help the examiners if you answer the questions in the order in which they are set.

Write your answers in one of the answer books provided.

DO NOT use the same answer book for Part 1 as for Part 2.

Please begin each Part 1 question on a new page.

Question 1

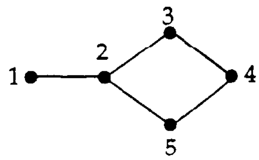
(a) Draw an appropriate graph to illustrate a job assignment problem in which five applicants, 1, 2, 3, 4, 5, apply for four jobs, a, b, c, d, as indicated in the following table.

applicant	job
1	b, c
2	a, c
3	b, c
4	a, c
5	d

(b) Determine the total number of ways in which the jobs can be filled by appropriate applicants.

Question 2

Consider the following labelled graph G.



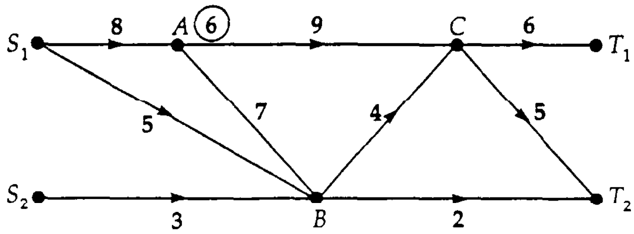
(a) Write down the adjacency matrix of G.

(b) Write down the degree sequence of G.

(c) Draw a simple connected graph with the same degree sequence as G which is *not* isomorphic to G.

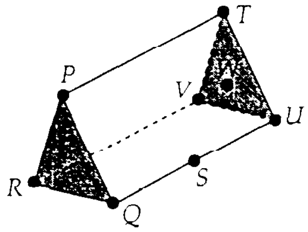
Question 3

Draw a basic network equivalent to the following network, in which the edge AB is undirected and the vertex A has capacity 6. (The numbers next to the arcs are capacities.)



Question 4

Consider the following figure, in which the point W lies on the face TUV.



Write down the convex hull of each of the following sets.

(a) $\{R, S\}$

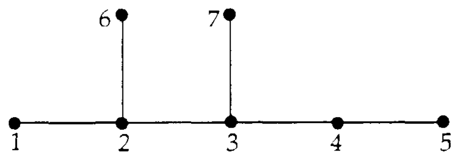
(b) $\{S, T, V\}$

(c) $\{S, T, U, V, W\}$

(d) $\{P, Q, S, T\}$

Question 5

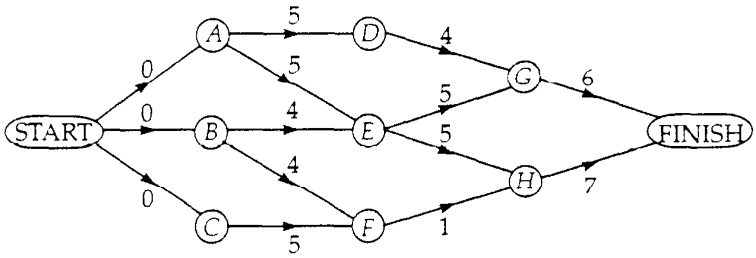
(a) Write down the Prüfer sequence which corresponds to the following labelled tree.



(b) Draw the labelled tree which corresponds to the Prüfer sequence (**4**, **3**, **5**, **3**, **4**).

Question 6

Consider the following activity network, in which the vertices represent activities and the numbers next to the arcs represent times (in days).



(a) Write down (without explanation):

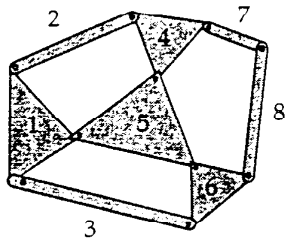
(1) the minimum completion time of the project if an unlimited number of workers are available;

(2) the corresponding critical path.

(b) Find the float of activity F.

Question 7

Consider the following kinematic system:

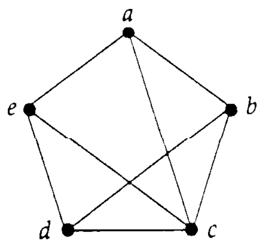


(a) Draw the interchange graph of the system.

(b) Calculate the mobility of the system when it is considered as a planar kinematic system containing only revolute pairs.

Question 8

Consider the following graph G.



(a) Determine the chromatic number $\chi(G)$.

(b) Determine the chromatic index $\chi'(G)$.

(Give a brief reason in each case.)

Question 9

A group of n men collectively know m women. The marriage condition is satisfied for all subsets of the n men except for one subset of five men. Show that exactly $n - 1$ men can marry women they know.

Question 10

Consider the following linear code:

$C = \{000000, 100001, 001100, 010010, 101101, 110011, 011110, 111111\}$.

(a) Determine whether the code C is cyclic.

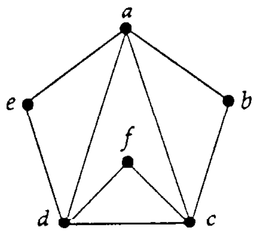
(b) Find the rate of the code C.

(c) Find a parity check matrix of the code C.

(d) Determine whether the dual of the code C is the same as the code C.

Question 11

Consider the following graph G.

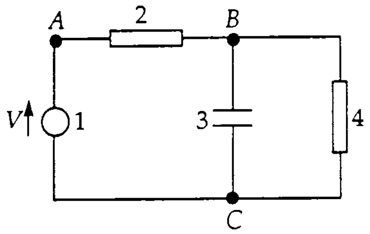


(a) Draw a depth-first search spanning tree of G, starting at the vertex labelled a.

(b) Draw a breadth-first search spanning tree of G, starting at the vertex labelled a.

Question 12

The following network consists of a battery of voltage V, a capacitor and two resistors.



(a) Draw an oriented graph representation and label it fully.

(b) Write down the component equations.

(c) Write down the state variable(s).

Question 13

Are there any balanced block designs with the following parameters?

(a) $v = 5$, $b = 10$, $r = 4$, $k = 2$, $\lambda = 1$;

(b) $v = 9$, $b = 10$, $r = 4$, $k = 3$, $\lambda = 1$.

In each case, justify your answer *either* by constructing a balanced design with the given parameters, *or* by explaining why such a balanced design cannot exist.

Part 2

Part 2 is divided into three sections. You should attempt **not more than FOUR** *questions from this part, including* **at least one from each section.**

Each question in this part is allotted **12 marks.**

Show all your working.

To help the examiners, please write the numbers of the questions you have attempted in Part 2 at the foot of the front cover of your answer book for this part.

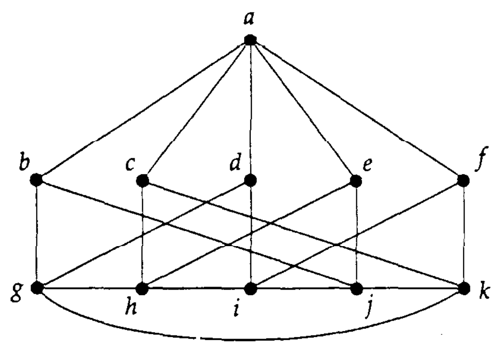
DO NOT use the same answer book as you used for Part 1.

Please begin each Part 2 question on a new page.

Section A Graphs

Question 14

Consider the following graph G.



(a) Show that G has no Hamiltonian cycle that includes the edges gh and hi. *(5 marks)*

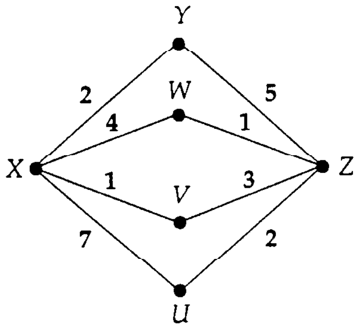
(b) Write down a Hamiltonian cycle containing the edges bg and gh. *(2 marks)*

(c) How many Hamiltonian cycles are there containing the edges bg and gh? *(1 mark)*

(d) Use the cycle method and the Hamiltonian cycle found in part (b) to determine whether G is planar. (Show all your working.) *(4 marks)*

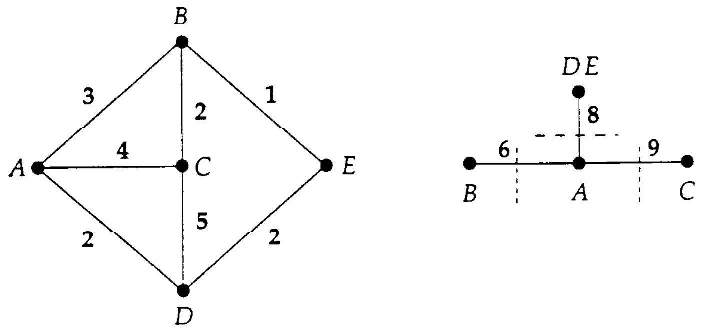
Question 15

(a) Perform the first iteration of the algorithm of Gomory and Hu to find the first branch of the cut tree corresponding to the following undirected network, using the vertex pair X, Y. (Show all your working.) *(3 marks)*



(The numbers next to the arcs are capacities.)

(b) (1) The algorithm of Gomory and Hu has been applied to the following undirected network, reaching the stage illustrated. Complete the final iteration of the algorithm and write down a cut tree for this undirected network. (Show all your working.) *(3 marks)*



(The numbers next to the arcs are capacities.)

(2) Use the cut tree found in part (1) to write down the values of the maximum flows between all pairs of vertices in this network. *(2 marks)*

(3) Draw the complete network which corresponds to the network used in parts (1) and (2). *(2 marks)*

(c) Theorem 4.2 of *Graphs 2* states: Let N be an undirected network with n vertices, and let $C(N)$ be the corresponding complete network. Then $C(N)$ has at most $n-1$ different capacities.

Explain briefly why $C(N)$ has at most $n-1$ different capacities. *(2 marks)*

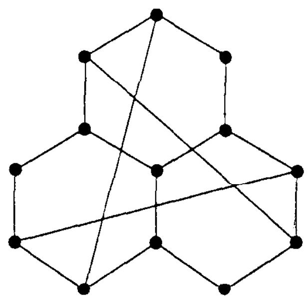
Question 16

Let G be a connected planar graph with n vertices, m edges and f faces.

(a) Suppose that a shortest cycle in G has length 6. Find an upper bound for the number of faces of G, in terms of the number of edges. *(2 marks)*

(b) Use Euler's formula for planar graphs to show that if a shortest cycle in G has length 6, then $m \leq \frac{3n-6}{2}$. *(3 marks)*

(c) Use part (b) to show that the following graph is non-planar. *(2 marks)*



(d) Show that if a shortest cycle in G has length k, where $k \geq 3$, then

$$m \leq \frac{k(n-2)}{k-2}.$$ *(3 marks)*

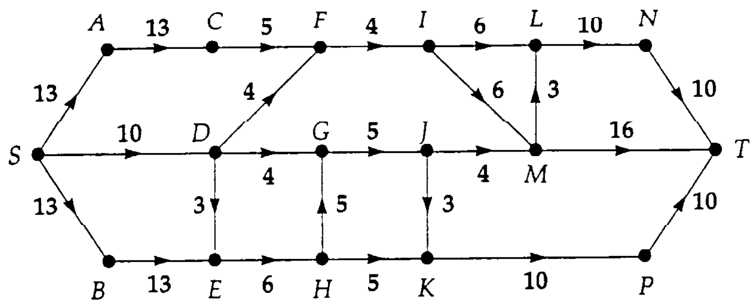
(e) Determine from part (d) that, for any planar bipartite graph with $n \geq 3$,

$$m \leq 2n - 4.$$ *(2 marks)*

Section B Networks

Question 17

Consider the following basic network, in which the number next to an arc is the capacity of the arc.



(a) Draw a diagram showing a maximum flow from S to T, clearly showing the flow along each arc.

Explain how you know that your flow is a maximum flow. *(5 marks)*

(b) For which arcs a of the basic network is it true that *decreasing* the capacity of the single arc a by one unit (while leaving all other capacities unchanged) *decreases* the value of the maximum flow from S to T?

(Explain your answer.) *(2 marks)*

(c) For which arc a of the basic network is it true that *increasing* the capacity of the single arc a by one unit (while leaving all other capacities unchanged) *increases* the value of the maximum flow from S to T?

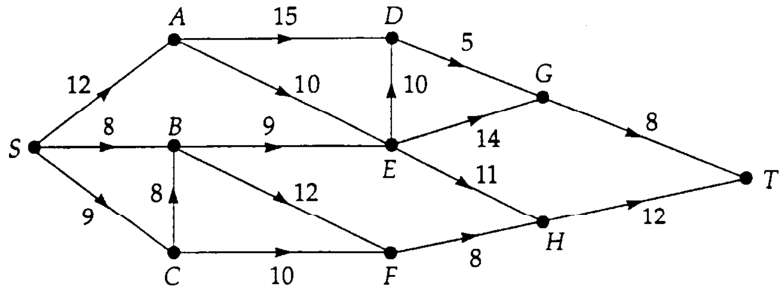
(Explain your answer.) *(2 marks)*

(d) What is the largest value of the maximum flow that can be obtained by increasing the capacity of a single arc?

(Explain your answer.) *(3 marks)*

Question 18

Consider the following weighted digraph.



(a) The *shortest* path algorithm is used to find the shortest path from S to T.

At a certain stage in the application of the algorithm,

- the vertices S, A, B, C, E have been assigned *potentials* 0, 12, 8, 9, 17, respectively;
- the vertices D, F, G, H have been assigned *labels* 27, 19, 31, 28, respectively.

Write down:

(1) the next vertex to be assigned a *potential* and the value of this potential;

(2) the next vertex label(s) to be assigned, once the potential in part (1) has been assigned;

(3) the remaining vertex potentials and the length of a shortest path;

(4) *all* the shortest paths from S to T. *(6 marks)*

(b) The *longest* path algorithm is used to find the longest path from S to T.

At a certain stage in the application of the algorithm,

- the vertices S, A, C, B, E, F have been assigned *potentials* 0, 12, 9, 17, 26, 29, respectively.

Write down:

(1) the remaining vertex potentials and the length of a longest path;

(2) *all* the longest paths from S to T. *(6 marks)*

Question 19

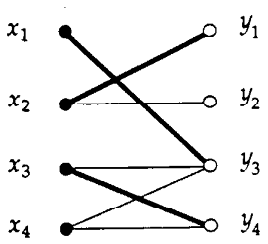
(a) The ability of each of four applicants x_1, x_2, x_3 and x_4 to do jobs y_1, y_2, y_3 and y_4 is represented by the costs in the following matrix, where a lower cost indicates a better ability.

	y_1	y_2	y_3	y_4
x_1	3	6	2	1
x_2	5	5	1	2
x_3	4	8	3	2
x_4	7	7	4	6

The Hungarian algorithm is to be used to find an allocation of applicants to jobs at minimum total cost. Construct the first revised cost matrix and the first partial graph. Do **not** proceed with the algorithm. *(3 marks)*

(b) In a different assignment problem, the following revised cost matrix and revised partial graph are obtained during the application of the Hungarian algorithm:

		2	0	0	-3
		y_1	y_2	y_3	y_4
2	x_1	3	2	0	4
1	x_2	0	0	4	12
5	x_3	1	3	0	0
5	x_4	1	3	0	0



In the revised partial graph, edges $x_1 y_3$, $x_2 y_1$ and $x_3 y_4$ are in the current matching.

Complete the application of the Hungarian algorithm to find a matching with the lowest possible total cost, and calculate the value of this lowest cost. (Show all your working.) *(9 marks)*

Section C   Design

Question 20

(a) (1) Sketch all possible arrangements of regular polygons around a vertex without gaps and in edge-to-edge contact, when only triangles, squares and octagons are allowed. Write down the code for each such arrangement. *(3 marks)*

(2) Show that there are only two arrangements of regular polygons around a vertex without gaps and in edge-to-edge contact, involving at least one octagon. *(4 marks)*

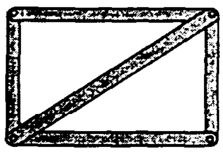
(b) A polyhedron has t triangular faces, s square faces and h hexagonal faces, and exactly four faces meet at each vertex.

(1) Write down expressions for the total number of vertices (v), edges (e) and faces (f) in terms of t, s and h. *(3 marks)*

(2) Show that $t = 2h + 8$. *(2 marks)*

Question 21

The following diagram shows a planar kinematic system consisting of five links interconnected by two binary revolute joints and two ternary revolute joints.



(a) By expanding both ternary joints into binary joints, derive and sketch the nine possible resulting kinematic systems. *(9 marks)*

(b) Derive and sketch the three non-isomorphic interchange graphs for the nine systems in part (a). *(3 marks)*

Question 22

(a) Show that $\{0, 1, 2, 6, 9\}$ is a perfect difference set (modulo 11). Hence write down a balanced block design Δ for 11 varieties in blocks of size 5. *(5 marks)*

(b) Write down the values of r and λ. *(1 mark)*

(c) Is the dual Δ^* of Δ a design? (Give a brief reason.) *(1 mark)*

(d) Let C and $\overline{C}$ be the codes whose codewords are the rows of the incidence matrices of Δ and its complement $\overline{\Delta}$ (respectively).

(1) Write down the parameters of the design $\overline{\Delta}$. *(2 marks)*

(2) Show that C and $\overline{C}$ can detect and correct the same numbers of errors. *(3 marks)*

[END OF QUESTION PAPER]