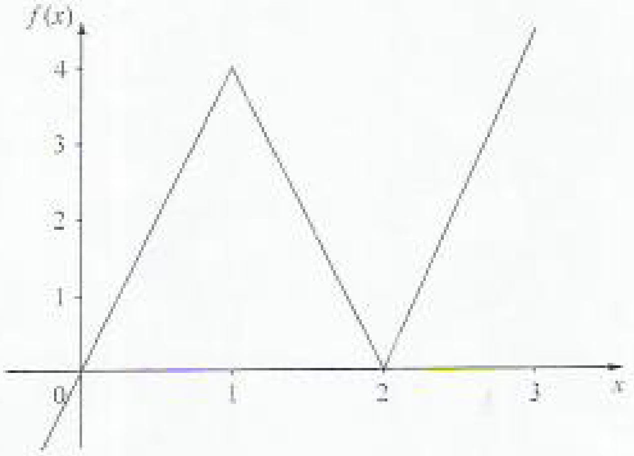
Question 5

(a) It follows from the definition of f that $f(x) = 0$ when $x = 0$ and $x = 2$.

Also, f has a local maximum at $x = 1$ and a local minimum at $x = 2$. We have $f(1) = 4$ and $f(2) = 0$. The graph of f is shown below.



4 marks

(b) It follows from the definition of f that the functions are

$$S_1(x) = x/4, \quad S_2(x) = -x/4 + 2 \text{ and } S_3(x) = x/4 + 2.$$

3 marks

(c) The mappings S_1, S_2 and S_3 are all contractions since $|S_i(x) - S_i(y)| = |x - y|/4$, for $x, y \in [0, 4]$ and $i = 1, 2, 3$.

Thus it follows from Theorem 9.1 of Falconer that there is a compact set F satisfying $F = S_1(F) \cup S_2(F) \cup S_3(F)$.

The functions S_1, S_2 and S_3 are in fact similarities with ratios $c_1 = c_2 = c_3 = 1/4$.

They also satisfy the open set condition (9.11) of Falconer with $V = (0, 4)$ (since the sets $S_1(V) = (0, 1)$, $S_2(V) = (1, 2)$ and $S_3(V) = (2, 3)$ are disjoint and all contained in V). (There are other possibilities for V here.)

Thus it follows from Theorem 9.3 of Falconer that $\dim_H F = s$, where s is given by

$$1 = \sum_{i=1}^{3} c_i^s = 3(1/4)^s.$$

Thus $s \log(1/4) = \log(1/3)$ and so $s = \log 3/\log 4$.

8 marks

(d) We begin by noting that F is invariant for f since $f(F) = f(S_1(F)) \cup f(S_2(F)) \cup f(S_3(F)) = F \cup F \cup F = F$.

We next note that, if $x < 0$, then $f(x) = 4x$ and so $f^n(x) \to -\infty$ as $n \to \infty$.

Also, if $x > 4$, then $f(x) > 2x$ and so $f^n(x) \to \infty$ as $n \to \infty$.

Thus any repeller for f must be contained in $[0, 4]$.

If $x \in [0, 4] \setminus F$, then, for some $k \in \mathbf{N}$, $x \notin S^k([0, 4])$ and so $f^k(x) \notin [0, 4]$.

Thus $f^n(x) \to \pm\infty$ as $n \to \infty$, for any $x \notin F$, and so F is indeed a repeller for f.

5 marks

(e) We denote the points of F by $x_{i_1, i_2, \ldots} = \bigcap_{k=1}^{\infty} S_{i_1} \circ S_{i_2} \cdots S_{i_k}([0, 4])$ with $i_j = 1, 2, 3$. Thus $f(x_{i_1, i_2, \ldots}) = x_{i_2, i_3, \ldots}$.

In part (c), we noted that $|S_i(x) - S_i(y)| = |x - y|/4$, for $x, y \in [0, 4]$ and $i = 1, 2, 3$. Thus $|x_{i_1, i_2, \ldots} - x_{i'_1, i'_2, \ldots}| \leq 4 \times 4^{-k}$ if $(i_1, \ldots, i_k) = (i'_1, \ldots, i'_k)$.