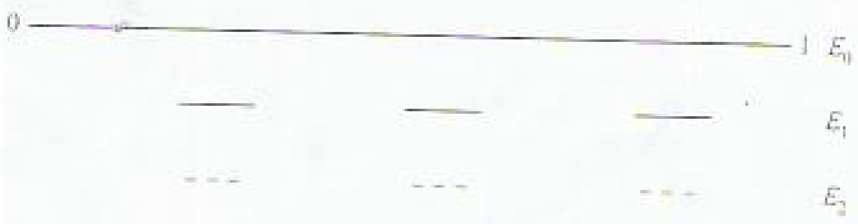
# M835 Solutions to the Specimen Examination Paper

## PART I

### Question 1

(a) The sets $E_0$, $E_1$ and $E_2$ are as shown in the following picture.




3 marks

(b) For each $k \in \mathbf{N}$, $E_k$ is non-empty and compact since it is the union of finitely many non-empty compact sets.

Since $E_{k+1} \subset E_k$, for each $k \in \mathbf{N}$, it follows that the sets $E_k$ form a decreasing sequence of non-empty compact subsets of $\mathbf{R}^2$ and hence $F$ is a non-empty compact set.

3 marks

(c) For each integer $k \geq 1$, the $n_k = 3^k$ intervals of length $\delta_k = 10^{-k}$ in $E_k$ form a cover of $F$.

Since $\delta_{k+1} = \frac{1}{10}\delta_k$, it follows from Proposition 4.1 that

$$\begin{aligned}\overline{\dim}_B F &\leq \overline{\lim}_{k\to\infty} \frac{\log n_k}{-\log \delta_k} = \overline{\lim}_{k\to\infty} \frac{3^k}{-\log(10^{-k})}\\ &= \overline{\lim}_{k\to\infty} \frac{k\log 3}{k\log 10} = \frac{\log 3}{\log 10}.\end{aligned}$$

Now note that $F = \bigcap_{k=0}^{\infty} E_k$, where, for each $k$, each interval of $E_{k-1}$ contains $m_k = 3$ intervals of $E_k$.

Also, the intervals in $E_k$ are separated by gaps of $\epsilon_k = 2 \times 10^{-k}$ so that $0 < \epsilon_{k+1} < \epsilon_k$, for each $k$, and so it follows from Example 4.6 of Falconer that

$$\begin{aligned}\dim_H F &\geq \underline{\lim}_{k\to\infty} \frac{\log(m_1 \dots m_{k-1})}{-\log(m_k \epsilon_k)} = \underline{\lim}_{k\to\infty} \frac{\log 3^{k-1}}{-\log(6 \times 10^{-k})}\\ &= \underline{\lim}_{k\to\infty} \frac{(k-1)\log 3}{-\log 6 + k\log 10} = \frac{\log 3}{\log 10}.\end{aligned}$$

Since $\dim_H F \leq \overline{\dim}_B F$, it follows that

$$\dim_H F = \dim_B F = \log 3 / \log 10.$$

11 marks

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