Now suppose that the sequence $(i_1, i_2, \ldots)$ is an infinite sequence with every finite sequence of 1s, 2s and 3s appearing as a consecutive block of terms. In this case, the orbit $\{f^k(x_{i_1,i_2,\ldots})\}$ is dense in F since, if $x_{i'_1,i'_2,\ldots} \in F$ and $q \in \mathbf{N}$, then there exists $k \in \mathbf{N}$ such that $(i'_1, \ldots, i'_q) = (i_{k+1}, \ldots, i_{k+q})$ and hence

$$|f^k(x_{i_1,i_2,\ldots}) - x_{i'_1,i'_2,\ldots}| = |x_{i_{k+1},i_{k+2},\ldots} - x_{i'_1,i'_2,\ldots}| \leq 4 \times 4^{-q}.$$

5 marks

[25 marks]

Question 6

(a) We use the fact that, if $g_a(z) = (\alpha^2 z^2 + 2\alpha\beta z + \beta^2 + c - \beta)/\alpha$, then $J(g_a) = h^{-1}(J(f_c))$, where $f_c(z) = z^2 + c$ and $h(z) = \alpha z + \beta$ — see p.204 of Falconer.

Here $g_a(z) = 3z^2 + iz + a$ and so $\alpha^2/\alpha = 3$, $2\alpha\beta/\alpha = i$ and $(\beta^2 + c - \beta)/\alpha = a$.

Thus $\alpha = 3$, $\beta = i/2$ and $c = 3a + \beta - \beta^2 = 3a + i/2 + 1/4$.

(i) It follows from Theorem 14.15 of Falconer that $J(f_c)$ is totally disconnected when $|c| > \frac{5+2\sqrt{6}}{4}$.

Thus $J(g_a) = h^{-1}(J(f_c))$ is totally disconnected when

$$|c| = |3a + i/2 + 1/4| > \frac{5 + 2\sqrt{6}}{4},$$

i.e. when

$$|a + i/6 + 1/12| > \frac{5 + 2\sqrt{6}}{12}.$$

(ii) As $a \to \infty$, $c = 3a + i/2 + 1/4 \to \infty$.

It follows from Theorem 14.15 of Falconer that, for large values of c,

$$\dim_H J(f_c) \sim 2\log 2/\log|c|.$$

Now $J(g_a) = h^{-1}(J(f_c))$. Since h^{-1} is a bi-Lipschitz transformation, it follows from Corollary 2.4 of Falconer that, for large values of a,

$$\begin{aligned}\dim_H J(g_a) = \dim_H J(f_c) &\sim 2\log 2/\log|3a + i/2 + 1/4| \\ &\sim 2\log 2/\log|3a|.\end{aligned}$$

12 marks

(b) (i) $f(4) = 4^2 - 12 = 4$ and so 4 is a fixed point of f.

Also $|f'(4)| = |2 \times 4| = 8 > 1$ so that 4 is a repelling fixed point of f.

Since $J(f)$ is closed and is the closure of the repelling periodic points of f, it follows that $4 \in J(f)$.

Also, $f(-4) = 4$ and so, since $J(f)$ is invariant under f and f^{-1}, it follows that $-4 \in J(f)$.

(ii) If $f(z) = z^2 - 12 \in [-4, 4]$, then $z^2 \in [8, 16]$.

Thus $z \in [-4, 4]$ i.e. $f^{-1}([-4, 4]) \subset [-4, 4]$.

It follows that $f(\mathbf{C} \setminus [-4, 4]) \subset \mathbf{C} \setminus [-4, 4]$ and hence $f^k(\mathbf{C} \setminus [-4, 4]) \subset \mathbf{C} \setminus [-4, 4]$, for each $k \in \mathbf{N}$.

Thus, by Montel's theorem, the family $\{f^k\}$ is normal on $\mathbf{C} \setminus [-4, 4]$ and hence $J(f) \subset [-4, 4]$.

(iii) If $z \in [-2, 2]$, then $f(z) = z^2 - 12 \in [-12, -8]$ and so, from part (b)(ii), $f(z) \notin J(f)$. Since $J(f)$ is invariant under f and f^{-1}, it follows that $z \notin J(f)$.

(iv) It follows from parts (ii) and (iii) that $J(f)$ is not connected and so -12 does not belong to the Mandelbrot set.

13 marks

[25 marks]

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