(c) From part (b) we have

$$\dim_H F \le \dim_B F = \log 4/\log 3.$$

Now let μ be the natural mass distribution on F so that each of the 4^k squares of side 3^{-k} in E_k carries a mass of 4^{-k}.

If $3^{-(k+1)} \le |U| < 3^{-k}$ for some $k \ge 1$, then U can intersect at most one of the squares in E_k and so

$$\mu(U) \le 4^{-k} = 4 \times 4^{-(k+1)} \le 4|U|^{\log 4/\log 3}.$$

It follows from the mass distribution principle 4.2 of Falconer that $\dim_H F \ge \frac{\log 4}{\log 3}$.

Thus $\dim_H F = \frac{\log 4}{\log 3}$. 8 marks

(d) If x and y are in the same component of E_k, for each $k \in \mathbf{N}$, then $|x - y| \le \sqrt{2}/3^k$, for each $k \in \mathbf{N}$, and hence $x = y$.

Thus, if x and y are distinct points in F, then there exists $k \in \mathbf{N}$ such that x and y belong to different components $E_{k,x}$ and $E_{k,y}$ of E_k.

Clearly, there exist disjoint open sets U and V such that $E_{k,x} \subset U$ whilst V contains the remaining $4^k - 1$ squares in E_k (including $E_{k,y}$).

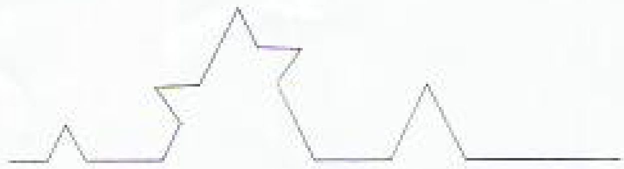
Thus $F \subset E_k \subset U \cup V$ whilst $x \in U$, $y \in V$ and $U \cap V = \emptyset$ so that x and y are in different connected components of F.

Thus F is indeed totally disconnected. 6 marks

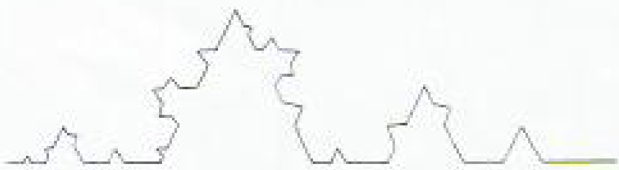
[25 marks]

Question 3

(a) The curve E_2 is shown below. (There are other possibilities here.)



The curve E_3 is shown below. (There are several possibilities here but only one that agrees with the choice of E_2.)



6 marks