Question 4

For $i = 1, 2$, let $S_i : \mathbf{R}^2 \to \mathbf{R}^2$ be the affine transformation defined by

$$S_i(t,x) = (t/2 + (i-1)/2,\ a_i t - 3x/4 + b_i).$$

Let F be the invariant set of S_1 and S_2.

(a) Determine values of a_i and b_i $(i = 1, 2)$ for which F is a self-affine curve passing through the points $(0,0)$, $(1/2, 1)$, $(1,0)$ with box dimension equal to 1.58 to two decimal places. [13]

(b) Let $E_0 = \{(x,y) : 0 \le x \le 1, y = 0\}$ and $E_{k+1} = \bigcup_{i=1}^{2} S_i(E_k)$, for each $k \in \mathbf{N}$. Sketch the first two stages, E_1 and E_2, in the construction of F, using the values of a_i and b_i $(i = 1, 2)$ determined in part (a).

(c) Modify S_1 and S_2 in such a way that the invariant set F is still a self-affine curve with E_1 remaining unchanged but with $\dim_B F = 1.85$ to two decimal places. [6]

Question 5

Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by

$$f(x) = \begin{cases} 4x, & x \le 1, \\ -4x + 8, & 1 < x < 2, \\ 4x - 8, & x \ge 2. \end{cases}$$

(a) Sketch the graph of f, identifying the values of x for which $f(x) = 0$ and any local maxima or minima of f. [4]

(b) Determine functions $S_1, S_2, S_3 : [0,4] \to \mathbf{R}$ such that $f(S_i(x)) = x$ for $i = 1, 2, 3$. [3]

(c) Show that there exists an invariant set, F, for S_1, S_2 and S_3 and determine $\dim_H F$. [8]

(d) Show that F is a repeller for f. [5]

(e) Show that there exists a point $x \in F$ such that the orbit $\{f^k(x)\}$ is dense in F. [5]

Question 6

(a) Let $g_a : \mathbf{C} \to \mathbf{C}$ be defined by $g_a(z) = 3z^2 + iz + a$, for each $a \in \mathbf{C}$. By using results in Falconer concerning the functions defined by $f_c(z) = z^2 + c$:

(i) determine a condition on a which ensures that $J(g_a)$ is totally disconnected;

(ii) find an asymptotic formula for $\dim_H J(g_a)$ as $a \to \infty$. [12]

(b) Let $f(z) = z^2 - 12$.

(i) Show that 4 is a repelling fixed point of f and hence show that $\pm 4 \in J(f)$.

(ii) Show that $f^{-1}([-4,4]) \subset [-4,4]$ and hence deduce from Montel's theorem that $J(f) \subset [-4,4]$.

(iii) Hence show that, if $z \in [-2,2]$, then $z \notin J(f)$.

(iv) Hence determine whether -12 belongs to the Mandelbrot set. [13]

[END OF QUESTION PAPER]