(d) For $k = 0, 1, 2, \ldots$ and $x, y \in [0, 1]$,

$$|f_k(x) - f_k(y)| = |x + kx^2 - y - ky^2|$$
$$= |x - y + k(x^2 - y^2)| = |x - y||1 + k(x + y)|$$

and so

$$|f_k(x) - f_k(y)| \leq (1 + 2k)|x - y|.$$

Thus f_k is a Lipschitz function, for $k = 0, 1, 2, \ldots$.

It follows from Corollary 2.4 of Falconer that, for $k = 0, 1, 2, \ldots$,

$$\dim_H f_k(F) \leq \dim_H F = \log 3 / \log 10.$$
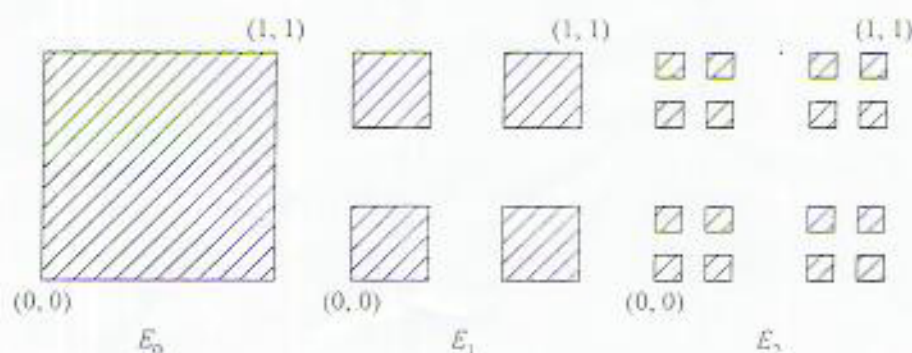
Also, by countable stability,

$$\dim_H \bigcup_{k=0}^{\infty} f_k(F) = \sup_{0 \leq k \leq \infty} \dim_H f_k(F).$$

Since $f_0(F) = F$, it follows that $\dim_H \bigcup_{k=0}^{\infty} f_k(F) = \log 3 / \log 10$.

8 marks

[25 marks]

Question 2

(a) The sets E_0, E_1 and E_2 are as shown in the following picture.

3 marks

(b) We use part (ii) of Equivalent definitions 3.1 of Falconer.

Let $N_\delta(F)$ denote the smallest number of cubes (i.e. squares) of side δ that cover F.

We will use the fact (see page 41 of Falconer) that, if $\delta_k = 3^{-k}$, then

$$\dim_B F = \lim_{k \to \infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k}.$$

provided this limit exists.

It follows from the construction of F that $N_{\delta_k}(F) \leq 4^k$ and so

$$\overline{\dim}_B F = \overline{\lim}_{k \to \infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \leq \overline{\lim}_{k \to \infty} \frac{\log 4^k}{\log 3^k} = \frac{\log 4}{\log 3}.$$

Now note that any square of side $\delta_k = 3^{-k}$ intersects at most four of the squares of side δ_k in E_k.

Since F meets each of the 4^k squares in E_k, it follows that

$$\underline{\dim}_B F = \underline{\lim}_{k \to \infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \geq \underline{\lim}_{k \to \infty} \frac{\log 4^k / 4}{\log 3^k}$$

$$= \underline{\lim}_{k \to \infty} \frac{(k - 1) \log 4}{k \log 3} = \frac{\log 4}{\log 3}.$$

So $\dim_B F = \log 4 / \log 3$.

8 marks

2