MS221 – 1999 Solutions
***** qns 1,3,4 no longer in syllabus**

Qn.1 (a)$x_1 \dots x_5 = \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}$
(b) (i) $d_1 \dots d_5$ =2,3,5,8,13,
(ii) $d_n = d_{n-1} + d_{n-2}$, d_6=21 d_7=34

Qn.2 (a) discriminant = 0^2-4.1.0 = 0
so conic is a parabola (n.b. discriminant is no longer in syllabus)
(b) (i) solve $x^2-2x-3=0$ => $(x-3)(x+1)=0$
=> x=3 or x=-1 so points are (3,0) and (-1,0).
(ii) $4(-y)^2= 4y^2$
(c) sketch is a parabola with its apex a minimum at (1,-2) and going through (0,-1.5), (3,0) and (-1,0).

Qn.3
(a) Solve 2p-(2.6-x)=x+4
ie. 2p-12+x=x+4 => p=8
(b) 2.5-(2.2-x)=10-4+x= 6+x
so transformation is trans_6

Qn.4 (a) $\begin{pmatrix} -4 \\ 4 \\ 4 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ (c) $\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix}$

Qn.5 (a) f is a continuous function, f(0)=0, f(1)=-1 so f is not increasing.
(b) Suppose $0 \le x_1 < x_2 \le 3$.
$g(x_1) - g(x_2) = 4x_1 - 4x_2 > 0$
this is true for any such x_1, x_2 so g is increasing.

Qn.6 (a) $(1+h)^6=$
$1+6h+15h^2+20h^3+15h^4+6h^5+h^6$
(b) Put h=-0.003
first 3 terms=1–0.018+0.000135
= 0.965525 = 0.982 to 3 d.p.
$|20h^3|$= 0.0000006, so this and subsequent terms are too small to count.

Qn.7 (a) solve $x^2-3x=x$, ie. $x^2-4x=0$
(b) fixed points are solutions of x(x-4)=0. that is 0 and 4.

Qn.8 (a) For eigenvalues solve
$(3-k)(2-k)-2=0$ that is
$k^2-5k+4=0$, $(k-1)(k-4)=0$
to give k=1 and k=4.

Eigenlines are given by
$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0, \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$
that is 2x+y=0, x=y

(b)Possible eigenvectors are $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ so take
$Q = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}, Q^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$.

Qn.9 (a) $\frac{1}{x^6}(x^3 \sec^2 x - 3x^2 \tan x)$
$= \frac{1}{x^4}(x \sec^2 x - 3\tan x)$ by quotient rule
(b) $\dot{g} = \frac{2t+4}{t^2+4t} = \frac{2(t+2)}{t^2+4t}$ by chain rule

Qn.10 (a) $I=\frac{1}{3}x\exp(3x) - \frac{1}{3}\int \exp(3x)dx$
$= \frac{1}{3}x\exp(3x) - \frac{1}{9}\exp(3x) + c$
(b) $u = 2x^2 + 3x$ $du=(4x+3)dx$
$I= \int \frac{2du}{u^2} = -\frac{2}{u} + c = -\frac{2}{2x^2+3x} + c$

Qn.11(a) $e^{-x}=1-x+(1/2!)x^2-(1/3!)x^3$
(b) $e^x=1+x+(1/2!)x^2+(1/3!)x^3$
so ch x = ½($e^x + e^{-x}$) =
$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots$
(c) sh x = $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots$

Qn.12 (a) $e^{2y}dy = xdx$
Integrate $\frac{1}{2}e^{2y} = \frac{1}{2}x^2 + c$
$2y = \ln(x^2+k)$
$y = \frac{1}{2}\ln(x^2+k)$

(b) At x=0,y=½, so ½=½ln k
=> ln k=1 => k=e and particular soln. is $y=½\ln(x^2+e)$.

Qn.13 (a) $z+\bar{z}=6$, $z\bar{z}=9-16i^2=25$
(b) $(x-(3+4i))(x-(3-4i)) = x^2-6x+25=0$

Qn.14 (a) (8+7+9+1)-(2+5+4+3) = 11
so x is divisible by 11.
(b) digit sum = 39, div. by 3 so x is div. by 3.
last 2 digits 28 = 0(mod 4)
so x is div. by 4.
x div. by 3 and 4 and so is div by 12.
(c) yes, 11 and 12 have no common factor so x is divisible by 132.

Qn.15 (a) r (b) p^{-1} =q, r^{-1} =r
(c) yes, because pq=qp=r

Qn.16(a)(B) is false – try n=1
(b) $n^2+n=n(n+1)$. one of any two consecutive integers is even so their product is even.

Qn.17 (a) (i) symmetry is about vertical axis through centre. Angles are multiples of $2\pi/n$ where n is the number of legs on the table.
(ii) if there were a marked spot not at the centre.
(b) A book - one plane divides the top half-pages from the bottom half-pages, another divides pages 1 to n/2 from pages (n/2)+1 to n.
(c) (i) Motif consists of 6 pentagons clustered round a point from which 6 equal sides radiate.
(ii) 5 successive rotations through $\pi/3$ about the vertex connecting the two equal sides.
(iii) Two translations, one to construct a strip, the other translates the strip to cover the plane.
(iv) The smallest angles in each pentagon are 60° since there are points surrounded by six of them.
The largest angles in each pentagon are 120° since there are points surrounded by three of them.

Qn.18 (a) (i)
$$\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}=\begin{pmatrix}4/5 & -3/5\\ 3/5 & 4/5\end{pmatrix}$$
which represents a clockwise rotation through α.
(b) (i) see p.57 in Handbook
$$\begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix}$$
$$=\begin{pmatrix}2\cos^2\alpha-1 & 2\sin\alpha\cos\alpha\\ 2\sin\alpha\cos\alpha & -2\cos^2\alpha+1\end{pmatrix}$$
$$=\begin{pmatrix}7/25 & 24/25\\ 24/25 & -7/25\end{pmatrix}$$
(ii) $M^{-1}=M$ since det(M)=-1 so transformation is again a reflection in 4y=3x.

(c)(i) LHS=$\begin{pmatrix}3/5 & -4/5\\ 4/5 & 3/5\end{pmatrix}\begin{pmatrix}6/5 & 8/5\\ -2/5 & 3/5\end{pmatrix}$=A
(ii) A^4=
$$\begin{pmatrix}3/5 & -4/5\\ 4/5 & 3/5\end{pmatrix}\begin{pmatrix}16 & 0\\ 0 & 1/16\end{pmatrix}\begin{pmatrix}3/5 & 4/5\\ -4/5 & 3/5\end{pmatrix}$$
$$=\begin{pmatrix}3/5 & -4/5\\ 4/5 & 3/5\end{pmatrix}\begin{pmatrix}9\frac{3}{5} & 12\frac{4}{5}\\ -\frac{1}{20} & \frac{3}{80}\end{pmatrix}$$
$$=\begin{pmatrix}0.6 & -0.8\\ 0.8 & 0.6\end{pmatrix}\begin{pmatrix}9.6 & 12.8\\ -0.05 & 0.0375\end{pmatrix}$$
$$=\begin{pmatrix}5.8 & 7.65\\ 7.65 & 10.26\end{pmatrix}$$ to 2 dec. places.
When P is large write 2^P=k so matrix product is
$$\begin{pmatrix}3/5 & -4/5\\ 4/5 & 3/5\end{pmatrix}\begin{pmatrix}k & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}3/5 & 4/5\\ -4/5 & 3/5\end{pmatrix}$$
$$=\begin{pmatrix}0.36k & 0.48k\\ 0.48k & 0.64k\end{pmatrix}$$
To observe the effect on y=x consider the image of $\begin{pmatrix}1\\1\end{pmatrix}$ which is
$$\begin{pmatrix}0.36k & 0.48k\\ 0.48k & 0.64k\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0.84k\\ 1.12k\end{pmatrix}$$

Ratio of items in this vector is 3:4 (nb. ratio of 13.45:17.91 in Mathcad is approx 3:4) so line y=x is transformed eventually into 3y=4x.

Qn.19

(a)Integrate by parts:

$A(x) = 4x\sin x - 4\int \sin xdx$

$= 4x\sin x + 4\cos x + c$

A(0)=0 so c=-4, so soln is

A(x) $= 4x\sin x + 4\cos x - 4$

so $A(\frac{\pi}{2}) = 4(\frac{\pi}{2}) - 4 = 2\pi - 4$

(b) A(x)=1 is equivalent to

$1 = 4x\sin x + 4\cos x - 4$

i.e. $x\sin x + \cos x - \frac{5}{4} = 0$

call this f(x)=0

$f(0)=-\frac{1}{4}$, $f(\frac{\pi}{2}) = (\frac{\pi}{2}) - \frac{5}{4} > 0$

f is continuous so there must be a root between 0 and $\pi/2$.

(c) N-Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \frac{x\sin x + \cos x - \frac{5}{4}}{(x\cos x)}$

If $x_0 = \pi/4$, $x_1 = \frac{\pi}{4} - \frac{\frac{\pi}{4}.\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{5}{4}}{\frac{\pi}{4}.\frac{1}{\sqrt{2}}}$

= 0.762949

(d) Ans to 4 dp=0.7629

Qn.20 (a) (i)

	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

all elements in table belong to {1,3,7,9} so G is closed.

1 is an identity element

Inverses of {1,3,7,9} are {1,7,3,9} respectively

Multiplication in general is associative, and thus so is $\times_{10}$

(b) (i) S(H)={I,H,P,Q} where

I =identity

P =reflection in x=y

Q =reflection in x=-y

H =half-turn

(ii) Inverses of {P,Q,H} are {P,Q,H}

(c) no – (G, $\times_{10}$) has two self inverse elements;

(S(H),o) has four.