MS221/P

The Open University

Second Level Course Examination 1999

Exploring Mathematics

Tuesday, 19 October, 1999 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can. You should attempt not more than **TWO** questions in Part II. Your answers to each part should be written in the answer books provided.

72% of the available marks are assigned to Part I and 28% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

Graph paper is available from the invigilator, if you feel it would assist you in answering questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified**.

Put your answer books together, with your signed desk record on top. Fix them all together with the fastener provided.

PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part I carries 72% of the available examination marks. Each question indicates how many of these marks are allocated to it.*

(iii) *You should record your answers to each question in the answer book(s) provided. You are strongly advised to show all your working, including any rough working.*

Question 1 – 4 marks

(a) Calculate the five terms x_1 to x_5 generated by the recurrence system

$$x_0 = 2,\ x_{n+1} = \frac{1}{x_n} + 1 \quad (n = 0, 1, 2, \ldots).$$

You should express each term as a fraction, as simply as possible. [2]

(b) The denominators of the fractions generated by the recurrence system in part (a) form a sequence d_n, $n = 1, 2, \ldots$.

(i) Write down the values of d_1, d_2, d_3, d_4 and d_5 which you found in part (a).

(ii) Explain how each term d_n may be calculated from terms earlier in the sequence and hence find the values of d_6 and d_7. [2]

Question 2 – 5 marks

The curve represented by the equation

$$x^2 - 2x - 2y - 3 = 0$$

is a conic section. It is symmetrical about the line $x = 1$ and crosses the y-axis at $(0, -\frac{3}{2})$.

(a) Find the value of the discriminant for the curve, and state what type of conic section is the curve. [1]

(b) Find the coordinates of the two points where the curve crosses the x-axis. [2]

(c) Use the information given and your answers to parts (a) and (b) to draw a rough sketch of the curve. [2]

Question 3 – 4 marks

This question concerns transformations of the line.

(a) Solve the following decomposition problem for the unknown transformation of the line ref_p:

$$\text{ref}_p\ \text{ref}_6 = \text{trans}_4.$$ [2]

(b) Find the transformation represented by $\text{ref}_5\ \text{ref}_2$. [2]

Question 4 – 5 marks

This question concerns the vectors **a** and **b**, where $\mathbf{a} = \begin{pmatrix} 5 \\ 0 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

(a) Find the vector joining the tip of **a** to the tip of **b**. [2]

(b) Hence find the vector joining the tip of **a** to a point one-quarter of the way from the tip of **a** to the tip of **b**. [1]

(c) Hence or otherwise find the vector which joins the origin to the point one-quarter of the way from the tip of **a** to the tip of **b**. [2]

Question 5 – 4 marks

(a) Consider the function $f: [0,\ 3] \to \mathbb{R}$ where $f(x) = 3x^2 - 4x$.

Show that f is not increasing. [2]

(b) Consider the function $g: [0,\ 3] \to \mathbb{R}$ where $g(x) = 4x - 5$.

Show that g is an increasing function. [2]

Question 6 – 5 marks

(a) Use the Binomial Theorem to obtain an expansion of $(1+h)^6$ as a sum of powers of h. [2]

(b) Use the expansion you obtained in part (a) to evaluate 0.997^6 correct to 3 decimal places. Explain briefly why you use the number of terms that you do. [3]

Question 7 – 3 marks

Consider the function $f(x) = x^2 - 3x$. $(x \in \mathbb{R})$.

(a) Write down the equation which the fixpoints of f satisfy. [1]

(b) Find the fixpoints of f. [2]

Question 8 – 6 marks

Let $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

(a) Show that the eigenvalues of **M** are 1 and 4, and find their corresponding eigenlines. [3]

(b) Find suitable matrices **Q** and $\mathbf{Q}^{-1}$ to express **M** in the form $\mathbf{QDQ}^{-1}$ where **D** is the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. [3]

Question 9 – 4 marks

Differentiate the following functions. In each case, state which of the principal rules of calculus you are using.

(a) $f(x) = \dfrac{\tan x}{x^3} \quad \left(0 < x < \dfrac{\pi}{2}\right)$ [2]

(b) $g(t) = \ln(t^2 + 4t) \quad (t > 0)$ [2]

Question 10 – 5 marks

(a) Using integration by parts, find the indefinite integral

$$\int x \exp(3x)\, dx.$$ [3]

(b) Using the substitution $u = 2x^2 + 3x$, or otherwise, find the indefinite integral

$$\int \frac{8x+6}{(2x^2+3x)^2}\, dx.$$ [2]

Question 11 – 4 marks

(a) Use an appropriate result from the Handbook to write down the Taylor series about $x = 0$ for the function $f(x) = e^{-x}$. [1]

(b) Hence write down the Taylor series about $x = 0$ for the function

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}).$$ [1]

(c) By differentiating term by term your answer to part (b), find the first three non-zero terms of the Taylor series about $x = 0$ for the function

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$ [2]

Question 12 – 5 marks

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = xe^{-2y}.$$ [3]

(b) Find the particular solution of this differential equation which satisfies the initial condition $y = \frac{1}{2}$ when $x = 0$. [2]

Question 13 – 5 marks

Let $z = 3 + 4i$.

(a) Calculate $z + \overline{z}$ and $z\overline{z}$, where $\overline{z}$ is the complex conjugate of z. [3]

(b) Find a polynomial with real coefficients whose roots are z and $\overline{z}$. [2]

Question 14 – 5 marks

In this question, you should use appropriate divisibility tests from those listed in the Handbook. If you use any other method, you will not be given any marks.

Let $x = 314\,957\,28$.

(a) Show that x is divisible by 11. [1]

(b) Show that x is divisible by 12. [3]

(c) Does it follow that x is divisible by 132? Briefly justify your answer. [1]

Question 15 – 3 marks

Consider the group $(G, *)$ whose Cayley table is given below.

$*$	p	q	r
p	q	r	p
q	r	p	q
r	p	q	r

(a) Which element is the identity element of $(G, *)$? [1]

(b) Write down the inverses of (i) p (ii) r. [1]

(c) Is $(G, *)$ Abelian? Briefly explain your answer. [1]

Question 16 – 5 marks

Consider the two statements (A) and (B) given below.

(A) $n^2 + n$ is always even; $n \in \mathbb{N}$.

(B) $n^2 + 2n$ is always even; $n \in \mathbb{N}$.

One of the statements (A) and (B) is true, and one is false.

(a) State which of (A) and (B) is false, and give a counterexample to demonstrate this. [2]

(b) Give a proof that the other statement is true. [3]

PART II

Instructions

(i) *You should attempt not more than* **TWO** *questions from this part of the examination.*

(ii) *Each question in this part carries 14% of the marks.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your working.*

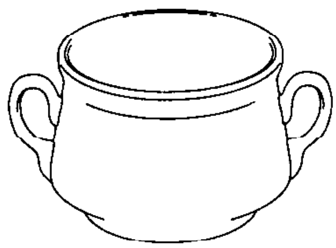
Question 17 – 14 marks

Below is an extract from Chapter A3 (pp. 54-55).

One important domain of application of transformations is in detecting or exploiting *symmetry*, which lies at the heart of our understanding of fundamental physics. Our material world comprises objects which exploit symmetry. The idea of symmetry is probably familiar to you intuitively. The human body is (approximately) symmetric in that, for example, the left hand is the mirror image of the right hand. The same goes for an archway, or a rose window in a cathedral. To define symmetry mathematically, we need to work a little harder.

Recall from the Introduction to this chapter that we defined an *isometry* to be a transformation that preserves distances (that is, lengths) and angles. Throughout the chapter we have been working with isometries – translations, rotations, reflections and composites of these (glide reflections) are all examples of isometries.

Generally speaking, the effect of an isometry on an object is detectable; it moves the object to a different place, turns it upside down, or whatever. However, there are exceptions: if you rotate a two-handled soup bowl by 180 degrees, you will normally be unable to observe the effect of the rotation – the bowl still looks the same. This is because it has 180-degree rotational *symmetry*.



Thus, being a little bit mysterious, we might say: 'A symmetry is an invisible isometry.' More formally:

> A **symmetry** of an object is an isometry which leaves the object setwise invariant.

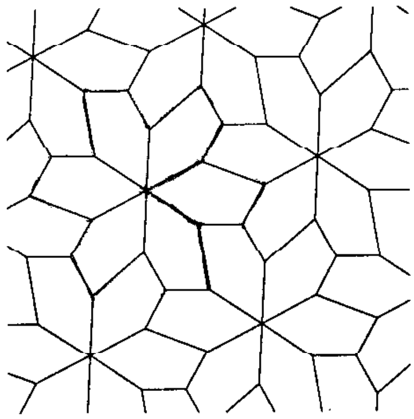
If you take a single transformation and apply it repeatedly to an object, preserving all the intermediate steps, the result is a pattern with symmetry.

(a) (i) Describe a circular table that has rotational symmetry, giving details of the angles of rotation involved. [2]

(ii) Under what circumstances would such a table have rotational symmetry only through 0°? [1]

(b) Describe clearly, or illustrate by means of a diagram, an everyday object that has reflective symmetry in two distinctively different planes, identifying the planes involved. [3]

(c) (i) For the following pattern (thought of as extending to cover the plane) find and clearly describe a motif made up of more than three pentagons, from which the whole pattern can be generated. [2]



(ii) Identify the transformations applied to a single pentagon which are required to produce the motif you have identified. [2]

(iii) What transformations would you apply to the whole motif in order to cover the plane? [2]

(iv) State any conclusions you can draw about the size of the angles of the single pentagon used to form the motif. [2]

Question 18 – 14 marks

(a) (i) Find the matrix which represents an anticlockwise rotation through the acute angle between the x-axis and the line $4y = 3x$. [2]

(ii) Write down the inverse of the matrix you found in part (a)(i) and describe the transformation which it represents. [2]

(b) (i) Find the matrix which represents a reflection in the line $4y = 3x$. (You will need to use the trigonometric identities in HB p.45.) [2]

(ii) Write down the inverse of the matrix you found in part (b)(i) and describe the transformation which it represents. [2]

(c) Let $\mathbf{A} = \begin{pmatrix} 1.04 & 0.72 \\ 0.72 & 1.46 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The following Mathcad screen has been set up to evaluate $\mathbf{A}^P$ and $\mathbf{A}^P\mathbf{w}$.

Matrix $A := \begin{pmatrix} 1.04 & 0.72 \\ 0.72 & 1.46 \end{pmatrix}$ Starting point $w := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ Number of iterations $P := 4$ $j := 1..P$

$A^P = \begin{pmatrix} 5.8 & 7.65 \\ 7.65 & 10.26 \end{pmatrix}$ Last calculated iteration $A^P \cdot w = \begin{pmatrix} 13.45 \\ 17.91 \end{pmatrix}$

(i) Show that $\begin{pmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{pmatrix} = \mathbf{A}$. [1]

(ii) Use part (c)(i) to show that Mathcad's result for A^P when $P = 4$ is correct to two places of decimals. [2]

(iii) Describe the effect of A^P on the line $y = x$ for large values of P. [3]

Question 19 – 14 marks

The area $A(x)$ under the graph of $y = 4x\cos x$ satisfies the initial-value problem

$$\frac{dA}{dx} = 4x\cos x \quad \left(0 < x \le \frac{\pi}{2}\right),$$
$$A(0) = 0.$$

(a) Find the solution of this problem, and evaluate $A(\pi/2)$. [5]

The remainder of this question concerns finding the value of x for which $A(x) = 1$.

(b) Show that the required value of x is a solution of the equation

$$x\sin x + \cos x - \tfrac{5}{4} = 0,$$

and verify that there is a root of this equation within the interval $[0, \pi/2]$. [4]

(c) Derive the Newton-Raphson formula required to find a root of the equation in part (b), and use this formula to find the value of the iterate x_1 corresponding to the starting value $x_0 = \pi/4$. [4]

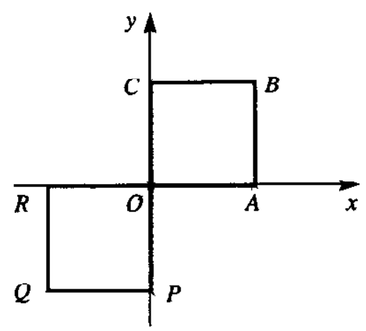
(d) Given that the calculated value of the next iterate, x_2, agrees with that of x_1 to four decimal places, write down to this accuracy the value of x for which $A(x) = 1$. [1]

Question 20 – 14 marks

(a) (i) Compile a Cayley table for the set $G = \{1, 3, 7, 9\}$ under the operation $\times_{10}$. [2]

(ii) Show that the set $G = \{1, 3, 7, 9\}$ is a group under the operation $\times_{10}$. [4]

(b) The plane set H consists of the two squares $OABC$ and $OPQR$ shown in the diagram below. The two squares are the same size.



(i) Using standard notation, write down the elements of the symmetry group $S(H)$ of H, giving a brief description of the geometric effect on H of each symmetry. [4]

(ii) State the inverse of each element in the group $(S(H), \circ)$. [2]

(c) Is $(S(H), \circ)$ isomorphic to $(G, \times_{10})$? Briefly justify your answer. [2]

[END OF QUESTION PAPER]