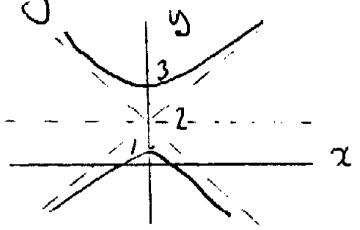
MS221 1998 Exam Answers

1. $1000x = 567.567\ldots = 567 + x$, $999x = 567$, $x = \frac{567}{999} = \frac{21}{37}$

2. (a) $x^2 - y^2 + 4y = 3$, if $y = 0$, $x = \pm\sqrt{3}$, crossing points on axis.
(b) No
(c) When x is very large, formula approaches $x^2 \doteqdot y^2$, hence $y \doteqdot \pm x$.
(d) Unchanged if we replace x by $-x$. Hence symmetric about y-axis
(e) When $y = 2$, $x^2 = -1$, hence no real value
(f) Rectangular hyperbola, coefficients of x^2 and y^2 equal & opposite — compare with standard form.



3. (i) $\text{Ref}_b \, \text{Ref}_a = \text{trans}_{2b-2a}$, $\text{Ref}_4 \, \text{Ref}_3 = \text{trans}_{2\times 4 - 2\times 3} = \text{trans}_2$
(b) (i) $2\times p - 2\times 5 = 4$, hence $p = 7$ (ii) $2\times 5 - 2\times p = 4$, hence $p = 3$

4. (a) $= b - a = \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}$ (b) $= \frac{1}{2}(a + b) = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

5. (a) No (b) $f(-3) = f(-1) = -2$

6. (a) ${}^jC_{j-k} = \dfrac{j!}{(j-k)!\,(j-(j-k))!} = \dfrac{j!}{(j-k)!\,k!} = {}^jC_k$ (b) $(1-2h)^4 = 1 - 8h + 24h^2 - 32h^3 + 16h^4$

7. (a) For $x < -2$, e^x becomes very small, & $f(x) \approx -6x$, which becomes large & positive. For $x > 4$, e^x becomes very large, with $|e^x| \gg |-6x|$, hence $f(x)$ becomes large and positive.
(b) Solution at about $x = \frac{1}{4}$. Start at $x = 0$ (exact solution is 0.204).

8 (a) From handbook $k^2 - (5+4)k + 5\times 4 - 2\times 1 = 0$, $k^2 - 9k + 18 = 0$, $k = 3$ or 6.
Substitute eigenvalues in $\begin{pmatrix} 5-k & 2 \\ 1 & 4-k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and solve

$k = 3$ $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives $x + y = 0$, $k = 6$: $\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives $x - 2y = 0$

(b) $Q = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, $Q^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$, and from these we can write down QDQ^{-1}.

9 (a) $f'(x) = \dfrac{3e^{3x}.\sin x - e^{3x}.\cos x}{\sin^2 x} = \dfrac{e^{3x}(3\sin x - \cos x)}{\sin^2 x}$ (quotient rule.)

(b) $u = \sec t$, $du/dt = \tan t \sec t$ $g(t) = \ln(u)$, $\dfrac{dg}{dt} = \dfrac{dg}{du}\cdot\dfrac{du}{dt} = \dfrac{1}{\sec t}.\tan t \sec t$

hence $dg/dt = g'(t) = \tan t$ (Used chain, or composite, rule)

10 (a) $\int x \ln(2x)\,dx = \frac{1}{2}x^2 \ln(2x) - \int \frac{1}{2}x^2 . \frac{1}{x}\,dx = \frac{1}{2}x^2\ln(2x) - \frac{1}{4}x^2 + C$

(b) $u = x^2+6x$, $du = (2x+6)\,dx$, Integral $= \int e^u \cdot \frac{1}{2}\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2+6x}+C$

11(a) $\frac{x^2}{1+x} = x^2\cdot\frac{1}{1+x} = x^2(1-x+x^2\ldots) = x^2-x^3+x^4-x^5\ldots\ldots$ $\quad(-1<x<1)$

(b) $\int \frac{x^2}{1+x}\,dx = \frac13 x^3 - \frac14 x^4 + \frac15 x^5 \ldots = \ln(1+x) - x + \frac12 x^2 + C$ $\quad(-1<x<1)$

12 (a) $\frac{dy}{y} = \frac{dx}{1+x^2}$, $\ln y = \arctan x + A$, $y = Ce^{\arctan x}$

(b) $\arctan 0 = 0$, hence $5 = Ce^0 = C$, and $y = 5e^{\arctan x}$

13(a) $z\bar z = (2-3i)(2+3i) = 13$

(b) $\frac{w}{z} = \frac{w\bar z}{z\bar z} = \frac{(1+2i)(2+3i)}{13} = -\frac{4}{13}+\frac{7}{13}i$

14 (a) $35 = 1\times 22 + 13$
$22 = 1\times 13 + 9$
$13 = 1\times 9 + 4$
$9 = 2\times 4 + 1$

$1 = 9 - 2\times 4 = 9 - 2(13-9)$
$= 3\times 9 - 2\times 13 = 3(22 - 1\times 13) - 2\times 13$
$= 3\times 22 - 5\times 13 = 3\times 22 - 5(35 - 1\times 22)$
$= 8\times 22 - 5\times 35$

hence inverse is 8

(b) Any multiple of 7 or 5 in $\mathbb{Z}_{35}$

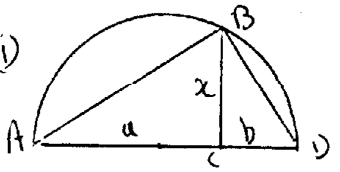
15.(a) 5 (b) (i) u (ii) p.

(c) $(G,*)$ is Abelian, order 6, 2 self-inverses, hence isomorphic to $(\mathbb{Z}_6, +_6)$

16 (a) $a(n)$ & $b(n)$ both necessary but not sufficient, $c(n)$ neither necessary nor sufficient, $d(n)$ not necessary but sufficient.

(b) $a(n) \wedge b(n)$

17(i)



Triangles are ABC, ABD, BCD.

(ii) $A\hat{B}D$ is angle subtended by diameter, hence is a right angle. Δ's ABC & ABD share $B\hat{A}C$, & both have a rt-angle, hence have all 3 angles the same and are similar. Δ's ABD and BCD share $B\hat{D}C$, and both have a rt angle, and so are similar. All three triangles are similar, as they have the same angles.

(iii) $\tan B\hat{A}C = x/a = \tan C\hat{B}D = b/x$, hence $x/a = b/x$, $x^2 = ab$.

(iv) Draw a circle radius 7, divide a diameter in ratio 5 : 2, ie a & b. Draw in perpendicular BC, which has length $\sqrt{10}$

18(a) Solve $x^2-1=x$, $x^2-x-1=0$, $x = \frac12(1\pm\sqrt5)$

(b) $f(f(x)) = (x^2-1)^2-1 = x^4-2x^2$

$f(f(x)) - x = x^4-2x^2-x = x(x^3-2x-1)$

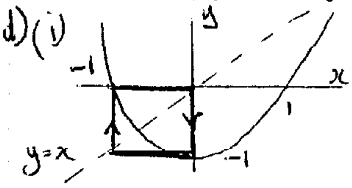
Consider given expression: $x(x+1)(x^2-x-1) = x(x^3-x^2-x+x^2-x-1)$
$= x(x^3-2x-1)$ which is the same as above.

when $f(f(x)) = x$, $f(f(x)) - x = 0$, $x(x+1)(x^2 - x - 1) = 0$.
Fix points are $0, -1, \frac{1}{2}(1 \pm \sqrt{5})$ ie last 2 the same as for $f(x)$.

(c)

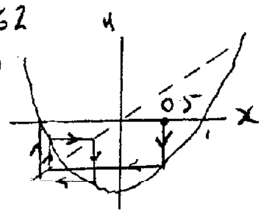
	x_1	x_2	x_3	x_4	x_5
$c=0$	0	−1	0	−1	0
$c=0\cdot5$	$0\cdot5$	$-0\cdot75$	$-0\cdot4375$	$-0\cdot8086$	$-0\cdot3462$

(d)(i)



settles immediately on a 2-cycle as shown

(ii)

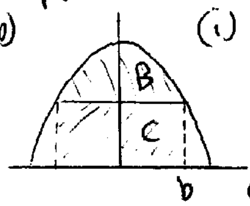


Converges on same 2-cycle as in (i)

19(a)(i) $A = 2\int_0^a \cos x \, dx = 2[\sin x]_0^a = 2\sin a$

(ii) $\int_{-\pi/2}^{\pi/2} \cos x \, dx = 2[\sin x]_0^{\pi/2} = 2$, half this area is 1, equating $2\sin a = 1$ gives $a = \arcsin(0\cdot5) = \pi/6$. From shape of graph, would expect more area nearer $x = 0$, hence a will be less than $\frac{1}{2} \times \frac{\pi}{2}$.

(b)



(i) B = total area $\int_{-b}^{b}$ less area C, and $C = 2b \times h = 2b \times \cos b$
hence $B = 2\sin b - 2b\cos b$
But half total area is 1, hence $B = 1 = 2\sin b - 2b\cos b$,
or $\sin b - b\cos b - \frac{1}{2} = 0$ as required.

(ii) $f'(b) = \cos b - \cos b + b\sin b = b\sin b$.
Hence N-R formula is $x_{n+1} = x_n - \dfrac{\sin x_n - x_n \cos x_n - \frac{1}{2}}{x_n \sin x_n}$

(iii) $b = 1\cdot2025$, $h = \cos b = 0\cdot3600$. Graph is wider at the bottom, hence h should be $< \frac{1}{2}$

20(a) $\langle 64, 0 \rangle$, (or more generally $\langle 64, 2k\pi \rangle$ where k is an integer)

(b) Roots are $2e^{\frac{2\pi k}{6} i}$, where $k = 0, 1, 2, 3, 4, 5$.

(c) The roots are in conjugate pairs, with $10\pi/6 = -2\pi/6$, $8\pi/6 = -4\pi/6$

Hence $(x^6 - 64) = (x-2)(x+2)(x - 2e^{2\pi/6\, i})(x - 2e^{-2\pi/6\, i})(x - 2e^{4\pi/6\, i})(x - 2e^{-4\pi/6\, i})$

$= (x-2)(x+2)(x^2 - 2(e^{\pi/3\, i} + e^{-\pi/3\, i})x + 4)(x^2 - 2(e^{2\pi/3\, i} + e^{-2\pi/3\, i})x + 4)$

But $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

hence $(x^6 - 64) = (x-2)(x+2)(x^2 - 4\cos\frac{\pi}{3}x + 4)(x^2 - 4\cos\frac{2\pi}{3}x + 4)$

$= (x-2)(x+2)(x^2 - 2x + 4)(x^2 + 2x + 4)$