MS221/Q

The Open University

Second Level Course Examination 1998

Exploring Mathematics

Tuesday, 20 October, 1998 2.30 pm – 5.30 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can. You should attempt not more than **TWO** questions in Part II. Your answers to each part should be written in the answer books provided.

72% of the available marks are assigned to Part I and 28% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

Graph paper is available from the invigilator, if you feel it would assist you in answering questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Put your answer books together, with your signed desk record on top. Fix them all together with the fastener provided.

PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part I carries 72% of the available examination marks. Each question indicates how many of these marks are allocated to it.*

(iii) *You should record your answers to each question in the answer book(s) provided. You are strongly advised to show all your working, including any rough working.*

Question 1 – 3 marks

By multiplying each side of the equation $x = 0.567\,567\,567\ldots$ by 1000, or otherwise, express the recurring decimal $0.567\,567\,567\ldots$ as a fraction. [3]

Question 2 – 6 marks

The curve represented by the equation

$$x^2 - y^2 + 4y = 3$$

crosses the y-axis at $y = 1$ and $y = 3$.

Answer the following questions about this curve.

(a) Does it cross the x-axis? If so, where? [1]

(b) Does it go through the origin? [1]

(c) When x is very large, what happens to y? Does it make any difference if x is very large and negative, or x is very large and positive? [1]

(d) Is it symmetrical? If so, about which line? [1]

(e) What value(s) does x take (if any) when $y = 2$? [1]

(f) What kind of conic section is this curve, and why? [1]

Question 3 – 6 marks

This question concerns transformations of the line.

(a) Find the translation represented by $\text{ref}_4\ \text{ref}_3$. [2]

(b) Solve the following decomposition problems for the unknown transformation of the line:

(i) $\text{ref}_p\ \text{ref}_5 = \text{trans}_4$ [2]

(ii) $\text{ref}_5\ \text{ref}_p = \text{trans}_4$ [2]

Question 4 – 3 marks

This question concerns the vectors $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$.

(a) Find the vector joining the tip of $\mathbf{a}$ to the tip of $\mathbf{b}$. [1]

(b) Find the vector which has its tip one-half of the way between the tips of $\mathbf{a}$ and $\mathbf{b}$. [2]

Question 5 – 3 marks

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as follows:

$$f(x) = \begin{cases} x+1 & (x < -2) \\ x-1 & (x \geq -2) \end{cases}$$

(a) State whether or not this function is one-one. [1]

(b) Justify your answer. [2]

Question 6 – 5 marks

(a) Show that ${}^{j}C_{j-k} = {}^{j}C_{k}$ [2]

(b) Use the Binomial Theorem to obtain an expansion of $(1 - 2h)^4$ as a sum of powers of h. [3]

Question 7 – 4 marks

Figure 1 shows a Mathcad screen set up to find a solution of the equation $f(x) = 0$ where

$$f(x) = \exp(x) - 6x.$$

using the root facility. It also shows the graph of f in the range specified.

$y := -2, -1.9 .. 4$

$f(x) := e^{x} - 6 \cdot x$

$x := 2$

$\text{root}(f(x), x) = 2.833$

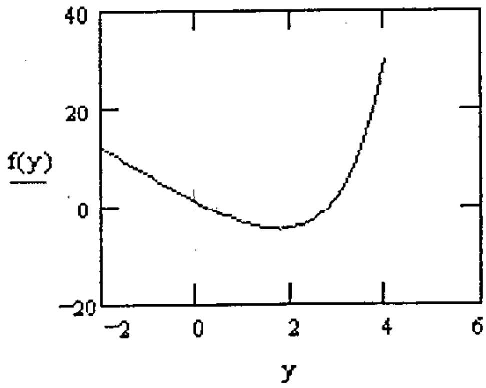

Figure 1

(a) It can be seen from the graph that there are only two solutions in the given range. Explain briefly why you might expect to find no further solutions when $x < -2$ or when $x > 4$. [2]

(b) From Figure 1, estimate the value of the solution not found by Mathcad and suggest a suitable starting value for x to find this. [2]

Question 8 – 6 marks

Let $\mathbf{M} = \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$.

(a) Show that the eigenvalues of $\mathbf{M}$ are 3 and 6, and find their corresponding eigenlines. [3]

(b) Express $\mathbf{M}$ in the form $\mathbf{QDQ}^{-1}$ where $\mathbf{D}$ is a diagonal matrix. [3]

Question 9 – 4 marks

Differentiate the following functions. In each case, state which of the principal rules of calculus you are using.

(a) $f(x) = \dfrac{e^{3x}}{\sin x} \quad (0 < x < \pi)$ [2]

(b) $g(t) = \ln(\sec t) \quad (-\frac{1}{2}\pi < t < \frac{1}{2}\pi)$ [2]

Question 10 – 5 marks

(a) Using integration by parts, find the indefinite integral

$$\int x \ln(2x)\,dx \quad (x > 0).$$ [3]

(b) Using the substitution $u = x^2 + 6x$, or otherwise, find the indefinite integral

$$\int (x+3)e^{x^2+6x}\,dx.$$ [2]

Question 11 – 5 marks

(a) Use an appropriate result from the Handbook to write down the Taylor series about $x = 0$ for the function

$$f(x) = \frac{x^2}{1+x}.$$ [2]

(b) By integrating term by term your answer to part (a), and comparing with another Taylor series given in the Handbook, find the indefinite integral

$$\int \frac{x^2}{1+x}\,dx \qquad (-1 < x < 1),$$

giving your answer in a form which does not feature a Taylor series. [3]

Question 12 – 4 marks

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{1+x^2} \quad (y > 0).$$ [3]

(b) Find the particular solution of this differential equation which satisfies the initial condition $y = 5$ when $x = 0$. [1]

Question 13 – 4 marks

Let $z = 2 - 3i$ and $w = 1 + 2i$.

(a) Calculate $z\overline{z}$, where $\overline{z}$ is the complex conjugate of z. [2]

(b) Write w/z in Cartesian form. [2]

Question 14 – 6 marks

(a) Use Euclid's algorithm to find the multiplicative inverse of 22 in $\mathbb{Z}_{35}$. [5]

(b) Give an example of a non-zero number in $\mathbb{Z}_{35}$ that has no multiplicative inverse. [1]

Question 15 – 4 marks

Consider the group $(G, *)$ whose Cayley table is given below.

$*$	p	q	r	s	t	u
p	u	r	t	p	s	q
q	r	s	p	q	u	t
r	t	p	u	r	q	s
s	p	q	r	s	t	u
t	s	u	q	t	r	p
u	q	t	s	u	p	r

(a) What is the identity element of $(G, *)$? [1]

(b) Write down the inverses of (i) r, (ii) t. [1]

(c) To which of the groups listed in the Handbook is $(G, *)$ isomorphic? Briefly justify your answer. [2]

Question 16 – 4 marks

The variable propositions $a(n)$, $b(n)$, $c(n)$ and $d(n)$, where $n \in \mathbb{N}$, have the meanings given below.

$a(n)$ means : n is divisible by 3

$b(n)$ means : n is divisible by 4

$c(n)$ means : n is divisible by 8

$d(n)$ means : n is divisible by 24

(a) For each of $a(n)$, $b(n)$, $c(n)$ and $d(n)$, say whether it is

(1) necessary but not sufficient,

(2) sufficient but not necessary,

(3) both necessary and sufficient, or

(4) neither necessary nor sufficient,

in order that n be divisible by 12. [2]

(b) Using a combination of the above propositions, give a condition that is necessary and sufficient for n to be divisible by 12. [2]

PART II

Instructions

(i) *You should attempt not more than* **TWO** *questions from this part of the examination.*

(ii) *Each question in this part carries 14% of the marks.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your working.*

Question 17 – 14 marks

Below is an extract from Chapter A2 (pp. 41–42).

> Given two lines of length a and b, their *mean proportional* is a line of length x with the property that $a/x = x/b$. In other words, $x = \sqrt{ab}$. Euclid's *Elements* (Book vi, Proposition 13) gives instructions on how to construct such a line: you draw a semicircle with diameter $a + b$ and erect a perpendicular where the two lengths meet. This is the required line.
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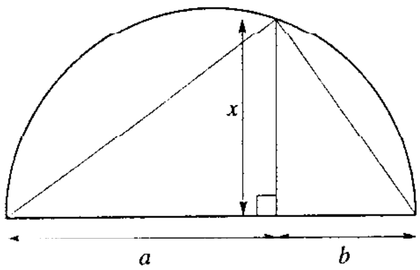
> 
>
>
> *Figure 2*
>
> It's not hard to see why this works, once you notice that all three triangles in the diagram are similar.

(i) By redrawing the diagram very roughly in your answer book and labelling it, explain which are the "three triangles in the diagram". [3]

(ii) Explain why the three triangles are similar. [3]

(iii) Describe the reasoning which leads you to conclude that the perpendicular in the diagram has the required property of being a mean proportional to a and b. [4]

(iv) This result seems to give a geometrical representation of an algebraic or arithmetical operation. By choosing plausible numbers for a and b, explain how you might use this construction to find the square root of 10. [4]

Question 18 – 14 marks

Let f be the function $f(x) = x^2 - 1 \quad (x \in \mathbb{R})$.

(a) Find the fixpoints of $f(x)$. [2]

(b) Write down an algebraic expression for $f(f(x))$. Hence show that

$$f(f(x)) - x = x(x+1)(x^2 - x - 1)$$

and give the fixpoints of $f(f(x))$. [4]

(c) Consider the iteration $x_{n+1} = f(x_n)$, $x_1 = c$.

Calculate the values x_2, x_3, x_4 and x_5 when

(i) $c = 0$;

(ii) $c = 0.5$. [2]

(d) Figure 3 shows a Mathcad screen which includes graphs of $f(x)$ and the line $y = x$, drawn on the same axes.

$x := -2, -1.9 .. 2$

$f(x) := x^2 - 1$

$x := -0.5$

$x := \text{root}(f(x) - x, x)$

$x = -0.618$

$\frac{d}{dx} f(x) = -1.235$

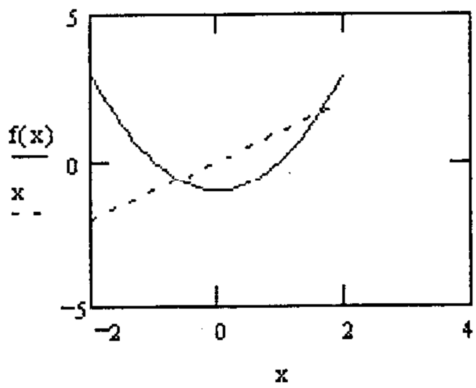


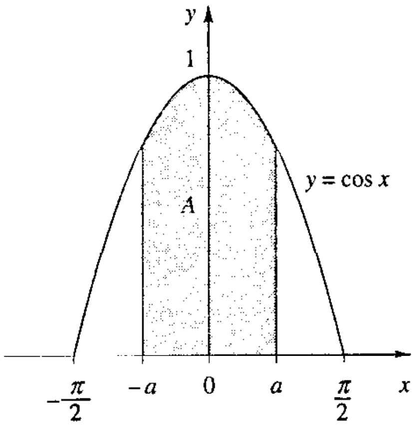
Figure 3

Describe the long-term behaviour of the iteration for the two starting values in part (c), and with reference to Figure 3 give a brief explanation of your answer. [6]

Question 19

This question concerns two different ways of dividing a region bounded by the graph of $y = \cos x$ and by the x-axis.

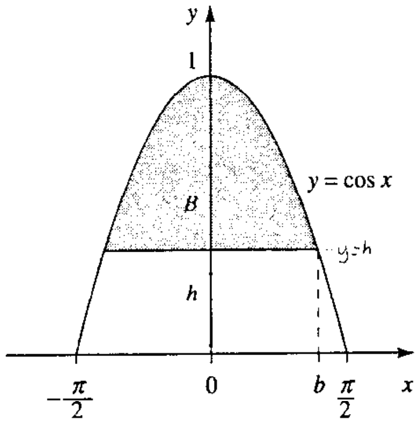
(a) The area A shown shaded in the figure below lies beneath the graph of $y = \cos x$, above the x-axis and between the vertical lines $x = -a$ and $x = a$.



(i) Find an expression for A in terms of a. [3]

(ii) Find the value of a for which A is equal to half of the area beneath the graph of $y = \cos x$ between $x = -\pi/2$ and $x = \pi/2$, and indicate why your answer appears sensible. [3]

(b) The area B shown shaded in the figure below lies beneath the graph of $y = \cos x$ and above the horizontal line $y = h$, where $h = \cos b$.



(i) Show that, if the area B is equal to half of the area beneath the graph of $y = \cos x$ between $x = -\pi/2$ and $x = \pi/2$, then b must satisfy the equation

$$\sin b - b\cos b - \tfrac{1}{2} = 0.$$ [3]

(ii) Derive the Newton–Raphson formula required to find the root b of this equation. [3]

(iii) When Mathcad was used to apply the Newton–Raphson method to the equation in part (b)(i), with starting value $b_0 = 1$, the following values were obtained respectively for b_1, b_2, b_3 and b_4:

1.236290169, 1.203154153, 1.202491204, 1.202490936.

Using these values, determine to four decimal places the required values of b and of h, and indicate why your answer for h appears sensible. [2]

Question 20

(a) Express the complex number 64 in polar form. [1]

(b) Calculate all the sixth roots of 64 and express them in exponential form. [6]

(c) Hence factorise the polynomial $x^6 - 64$ into polynomial factors with real coefficients, where the factors are either linear or quadratic. [7]

[END OF QUESTION PAPER]