The Open University

MS221/M

Second Level Course Examination 1997
Exploring Mathematics

Monday, 20 October, 1997 2.30 pm – 5.30 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can. You should attempt not more than **TWO** questions in Part II. Your answers to each part should be written in the answer books provided.

72% of the available marks are assigned to Part I and 28% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

Graph paper is available from the invigilator, if you feel it would assist you in answering questions.

At the end of the examination

Check that you have written your name, personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part I carries 72% of the available examination marks. Each question indicates how many of these marks are allocated to it.*

(iii) *You should record your answers to each question in the answer book(s) provided. You are strongly advised to show all your working, including any rough working.*

Question 1 – 3 marks

By multiplying each side of the equation $x = 0.345\,345\,345\ldots$ by 1 000 or otherwise, express the recurring decimal 0.345 345 345... as a fraction. [3]

Question 2 – 6 marks

Two curves are represented by the equations

$(C1)$ $x^2 = \frac{y}{2} + 1$ and

$(C2)$ $x^2 + \frac{y^2}{4} = 1.$

(a) On the same diagram, sketch the curves $(C1)$ and $(C2)$, identifying the points where they cross the axes and indicating where they intersect each other. [4]

(b) What type of curve does each of equations $(C1)$ and $(C2)$ represent? [2]

Question 3 – 6 marks

This question concerns transformations of the line.

(a) (i) What is the result of composing trans_6 and trans_4? [1]

(ii) What is the result of composing ref_6 with itself? [1]

(b) Solve the following decomposition problems for the unknown transformation of the line, by finding the value of p in each case.

(i) $\text{ref}_p\text{ref}_4 = \text{trans}_6$ [2]

(ii) $\text{ref}_4\text{ref}_p = \text{trans}_6$ [2]

Question 4 – 3 marks

Find the vector which has its tip two-thirds of the way between the tips of the vectors **b** and **a**, (that is closer to **a** than to **b**) where $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$. [3]

Question 5 – 3 marks

Consider the function $f : [-1, 1] \to \mathbb{R}$, where $f(x) = x^2 - 3x$.

Decide and state whether this function is increasing or not increasing, and justify your answer. [3]

Question 6 – 6 marks

Figure 1 shows a grid connecting A to B with five corridors across and six corridors down.

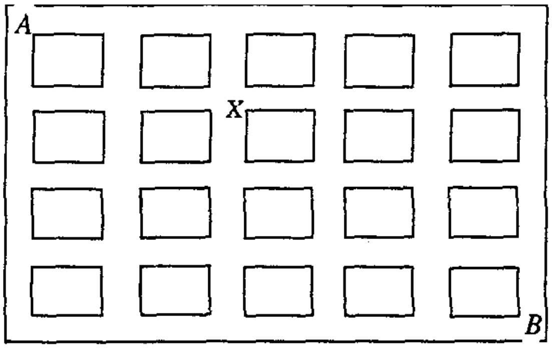


Figure 1

(a) How many different routes are there through the grid from A to B (where each route can be traversed only rightwards and downwards)? [2]

(b) What proportion of the total number of such routes from A to B pass through X? [4]

Question 7 – 3 marks

Find the fixpoints of the function $f(x) = x^2 - 4x + 6$ $(x \in \mathbb{R})$.

Question 8 – 6 marks

The matrix $\mathbf{M} = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$ has eigenvalues 2 and 1. The corresponding eigenvectors are $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, for eigenvalue 2; and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, for eigenvalue 1.

(a) Express $\mathbf{M}$ in the form $\mathbf{QDQ}^{-1}$, where $\mathbf{D}$ is a diagonal matrix. [3]

(b) Using your result from (a), calculate $\mathbf{M}^{10}$. You may express your answer as the product of three matrices. [3]

Show your working: a value of $\mathbf{M}^{10}$ *obtained from a calculator will receive no credit.*

Question 9 – 5 marks

Differentiate the following functions. In each case state which of the principal rules of calculus you are using.

(a) $f(x) = \sin(\ln(2x)) \quad (x > 0)$. [2]

(b) $g(x) = \sqrt{\dfrac{x}{\sin x}} \quad (0 < x < \pi)$. [3]

Question 10 – 5 marks

(a) Find the indefinite integral $\displaystyle\int \frac{1+x}{\sqrt{x}}\,dx \quad (x \neq 0)$. [2]

(b) Find the indefinite integral $\displaystyle\int \frac{\cos x}{3+\sin x}\,dx$ using the substitution $u = 3 + \sin x$. [3]

Question 11 – 3 marks

Using standard results from MS221 Handbook C, determine the Taylor series expansion about zero for the following function, as far as the fourth non-zero term.

$$f(x) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right).$$ [3]

Question 12 – 5 marks

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = y\cos x \quad (y > 0).$$ [3]

(b) Find the particular solution of this differential equation which satisfies the initial condition $y = 1$ when $x = 0$. [2]

Question 13 – 4 marks

(a) Express the complex number $z = \sqrt{3} + i$ in the polar form $\langle r, \theta \rangle$, where θ is the principal value of the argument of z. [2]

(b) Find, in polar form, three complex numbers which satisfy the equation $w^3 = \sqrt{3} + i$. [2]

Question 14 – 4 marks

(a) Find a number x in $\mathbb{Z}_{12}$ such that $x \times_{12} 7 = 5$. [2]

(b) Give an example of a non-zero number x in $\mathbb{Z}_{12}$ which has no multiplicative inverse. [2]

Question 15 – 5 marks

The Cayley table of a group $(G, *)$ is given below.

$*$	p	q	r	s
p	s	r	p	q
q	r	s	q	p
r	p	q	r	s
s	q	p	s	r

(a) What is the identity element of $(G, *)$? [1]

(b) Is $(G, *)$ Abelian? Briefly explain your answer. [1]

(c) Write down all the self-inverse elements of $(G, *)$. To which, if any, of the groups tabulated in Handbook D, is $(G, *)$ isomorphic? Briefly explain your answer. [3]

Question 16 – 5 marks

Here are two statements about integers p and q, only one of which is true.

(A) If both p and q are even, then the product pq is even.

(B) If a product pq is even, then both p and q are even.

(a) Which of these is the false statement? [1]

(b) Prove that the statement you have identified in (a) is false. [2]

(c) What is the name of the style of proof which you have used in (b)? [2]

PART II

Instructions

(i) *You should attempt not more than* **TWO** *questions from this part of the examination.*

(ii) *Each question in this part carries 14% of the marks.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your working.*

Question 17

Below is an extract from Chapter A1 (pages 37–38).

Nesting pentagons

The connections between ϕ and the regular pentagon go further, however. When the diagonals are drawn another, smaller, regular pentagon appears in the middle. This new pentagon can have *its* diagonals drawn in and produce another, even smaller, pentagon, and so on ... (see Figure 2).

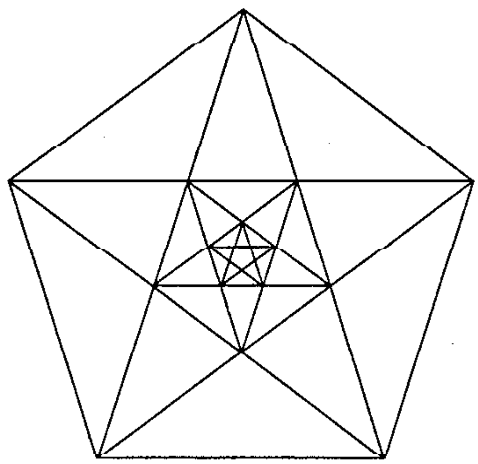


Figure 2

Each of these smaller pentagons is a scaled-down version of the previous one, with the same scale factor each time (since they are produced by the same method).

But the process can also be imagined in reverse. The original pentagon can be thought of as the middle pentagon inside a larger pentagon, and that pentagon can be thought of as being in the middle of an even larger one, ... Because the diagonals form another regular pentagon inside, these larger pentagons are each produced by extending the edges of the previous pentagon. [...] So there is a 'doubly infinite' set of pentagons nesting inside one another. What are the lengths of the sides and diagonals of these new pentagons? Once the scale factor of enlargement is known these lengths can be determined. To find this scale factor, we look at the connection between the lengths of a diagonal in the original pentagon and a diagonal in the next larger one.

The original pentagon, call it P_0, has side length 1 and diagonal length ϕ.

(a) Explain why the scale factor of the nested pentagons is 'the same scale factor every time'. [2]

(b) Express this scale factor in terms of ϕ. Explain how the result is reached. [4]

(c) There are a number of other ways of envisaging the number ϕ, the diagonal length of the original pentagon. Indicate two of these. How does the connection arise with the pentagon? [5]

(d) There are also some rather dubious connections made between the number ϕ and other things, for example, the architecture of the Parthenon and the structure of Beethoven's fifth symphony. Indicate why such connections should be regarded sceptically. [3]

Question 18

The function F and algorithm $BINCFTS$ below were introduced in Chapter B2. Suppose that **A** is any matrix with only 1 row. Then $F(\mathbf{A})$ is the matrix obtained from **A** by the following procedure:

Form a matrix **B** by inserting one extra 0 to the right of **A**.

Form a matrix **C** by inserting one extra 0 to the left of **A**.

Add the matrices **B** and **C**. This sum is $F(\mathbf{A})$.

```
alg BINCFTS
(input n ∈ ℕ)
power := 0
matrix := [1]
loop while power < n
     power := power + 1
     matrix := F(matrix)
endloop
(output is matrix)
```

(a) Trace the algorithm $BINCFTS$ when $n = 5$ is input. Check your output by calculating $(1 + x)^5$ using the Binomial Theorem. [4]

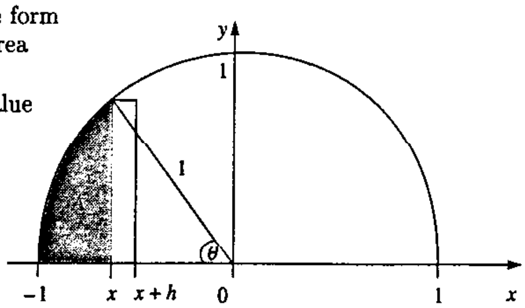
(b) Suppose you want an algorithm that calculates the coefficients in the expansion of $(1 + 2x)^n$: yielding, for example, $[1, 4, 4]$ when 2 is input, because $(1 + 2x)^2 = 1 + 4x + 4x^2$.

(i) Suppose that **M** is the matrix of coefficients in the expansion of $(1 + 2x)^k$. Describe informally how to obtain from **M** the matrix of coefficients in the expansion of $(1 + 2x)^{k+1}$. Hence describe how to modify the definition of F to obtain the required algorithm. Explain briefly why $BINCFTS$ itself does not need changing, so long as the definition of F is modified. [7]

(ii) Trace your algorithm when $n = 4$ is input, and check your output using the Binomial Theorem. [3]

Question 19

The figure on the right shows a semicircle of radius 1 whose centre is at the origin. The shaded region within the semicircle is bounded by a line with equation of the form $x =$ constant, and the corresponding area of the region is denoted by $A = A(x)$. In terms of the angle θ marked, this value of x is given by $x = -\cos\theta$.



(a) By regarding the difference

$$A(x+h) - A(x)$$

as being approximately equal to the area of the narrow rectangle shown on the figure, and taking an appropriate limit, show that

$$\frac{dA}{dx} = \sin\theta.$$ [2]

(b) By using the Chain Rule, or otherwise, show that A satisfies the conditions

$$\frac{dA}{d\theta} = \sin^2\theta,$$
$$A = 0 \text{ when } \theta = 0.$$ [3]

(c) (i) Find the solution of the initial-value problem stated in part (b). [3]

(ii) Use your solution to verify that the area of the whole semicircle is $\frac{1}{2}\pi$. [1]

The rest of the question concerns finding the value of x for which the corresponding value of A is two-thirds the area of the whole semicircle.

(d) (i) Show that the corresponding value of θ satisfies the equation

$$\theta - \tfrac{1}{2}\sin(2\theta) - \tfrac{2}{3}\pi = 0.$$ [1]

(ii) Write down the Newton-Raphson formula required to find the root θ of this equation. [2]

(e) When Mathcad was used to carry out this iteration, with starting value $\theta_0 = 2$, the following values were obtained for θ_n, where $n = 1, 2, 3$ respectively:

1.82825432, 1.83889943, 1.83892982.

(i) Give the solution to the equation in part (d)(i), as accurately as permitted by these figures, and explain why it is justified to claim this accuracy. [1]

(ii) Find the corresponding value of x, correct to three decimal places. [1]

Question 20

A cipher on $\mathbb{Z}_{26}$ is defined by the function $f(m) = 11 \times_{26} m$.

(a) Using Euclid's algorithm, determine the multiplicative inverse of 11 in $\mathbb{Z}_{26}$. [8]

(b) A message using the correspondence $1 = A, 2 = B, \ldots, 26 = Z$ is enciphered using f to give the cipher text

10, 11, 20, 20, 15, 22, 21, 16, 12, 10, 18, 11, 15.

What was the message? [6]

[END OF QUESTION PAPER]