MS221 – 2004 Solutions

Qn.1 (a) $u_2 = 0$

$$u_n = 4^n(1-\tfrac{1}{2}n) \quad (n = 0,1,2,..)$$

Qn.2 (a) eqn. is $\frac{x^2}{36} - \frac{y^2}{9} = 1$

which is a hyperbola in standard form.

(b) (i) $e^2 = 1+9/36 = 5/4$ so $e = \frac{\sqrt{5}}{2}$

(ii) Foci are $(\pm 3\sqrt{5}, 0)$

(iii) asymptotes are $y = \pm x/2$,

points of intersection are (6,0) and (-6,0).

Qn.3. (a) $t_{7,-5}$ (b) $(x + 11, y - 7)$

(c) $(y + 5) = 2(x - 7) + 3$,

that is $y = 2x - 16$

Qn.4 (a) $\sin A = \frac{\sqrt{5}}{3}, \cos A = 2/3$ (b) $\frac{4\sqrt{5}}{9}$

Qn.5 (a) 126 (b) -489888

Qn.6 (a) $A=\begin{pmatrix} 6 & -3 \\ -8 & 4 \end{pmatrix}$ $B=\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$

(b) Determinant of $A = 0$, line is $y=-4x/3$

(c) $\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 6 & -3 \\ -8 & 4 \end{pmatrix} = \begin{pmatrix} 14 & -7 \\ -4 & 2 \end{pmatrix}$

(d) a flattening onto the line $y=-2x/7$

Qn.7 (a) eigenvalues are 3 and 4.

(b) $D=\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ $P=\begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$

(n.b. the above answer is not unique, another possibility is

$D=\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ $P=\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$)

Qn.8 (a) is correct. (b) is wrong because for +ve e-values, points are on the same side of the line as x_0.

(c) is wrong because distances along the line are in geometric progression. (see also p.63 in Handbook).

Qn.9 (a) $\sec^2(\sqrt{x}) \big/ 2\sqrt{x}$

(b) $\frac{e^{4x}(4\cos x + \sin x)}{\cos^2 x}$

Qn.10 (a) $f(1/2)=1/2 - \ln(1/2) < 0$; $f(1) = 1 > 0$ so $f(x) = 0$ has a solution in (0,1)

(b) $f'(x) = 1 + \frac{1}{x}$, hence $x - f(x)/f'(x) =$

$$x - \frac{x(x+\ln x)}{1+x} = \frac{x(1-\ln x)}{1+x}$$

from which the N-Raphson formula follows.

(c) $\frac{0.6(1-\ln 0.6)}{1.6} = 0.567$

Qn.11 (a) Integral = $2e^{x/2}(x+3) - \int 2e^{x/2}dx$

$= 2e^{x/2}(x+3) - 4e^{x/2} + c = 2e^{x/2}(x+1) + c$

(b) $u=\ln x$; $du/dx = 1/x$;

$I = \int \frac{du}{1+u^2} = \arctan u = \arctan(\ln x) + c$

Qn.12 (a) $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$

(b) $\frac{1}{24}x^3 - \frac{1}{16}x^4$

(c) $\frac{0.000008}{24} - \frac{0.00000016}{16} = 3.2 \times 10^{-7}$

to 2 s.f.

Qn.13 (a) $\sqrt{74}$, 7 + 5i

(b) $\frac{3+2i}{7-5i} = \frac{11+29i}{74}$

Qn.14 (a) 9

(b) any even number in 0,..12

Qn.15 (a) $\{ r_0, r_p, q_0, q_{p/2} \}$ = {I,H,X,Y} say

(b)

	I	H	X	Y
I	I	H	X	Y
H	H	I	Y	X
X	X	Y	I	H
Y	Y	X	H	I

(c) group is isomorphic to the group of symmetries of the rectangle because the same reflections and rotation restore a rectangle to its original position (or alternatively because all the four elements are self-inverse).

Qn.16 Add $(n+1)^2$ to the given r.h.side, then factorise out (n+1) to give

1/6 (n+1) [n(2n+1)+6(n+1)]

The expression in the square brackets is $2n^2+7n+6$ which factorises to (n+2)(2n+3), so the new sum is 1/6(n+1)(n+2)(2n+3) which is the 'n+1' form.

$1/6(1.2.3) = 1^2$ so result is true for n=1. Therefore it is true for all n.

Qn.17 (a) Complete the squares:
$4(x + 5)^2 + 9(y - 1)^2 = 36$ and divide by 36.

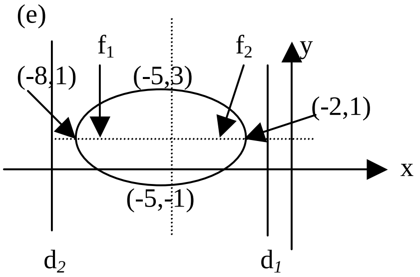
(b) t-5,1 $e = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$

(c) $ae = \sqrt{5}$ so foci are $(-5 \pm \sqrt{5}, 0)$

(d) Directrices are $x = -5 \pm \frac{9}{\sqrt{5}}$

that is $x = -5 \pm \frac{9\sqrt{5}}{5} = \frac{1}{5}(\pm 9\sqrt{5} - 25)$

(e)



axes of symmetry (dotted) are x = -5, y= 1

(f) x = 3 cos t - 5 ; y = 2 sin t +1

Qn.18 (a) (i) Solve g(x) = 0
where $g(x) = x^2 + 2.5x$ to give
fixed points 0 and -2.5
(ii) f'(x) = 2x + 3.5
for x = 0 | f'(x)| >1
for x = -2.5 | f'(x)| >1
so both fixed points are repelling

(b) (i) $f(f(x)) = (x^2 + 3.5x)^2 + 3.5(x^2 + 3.5x)$
$= x^4 + 7x^3 + 15.75x^2 + 12.25x$

(ii) f(f(x)) - x = x(x + 2.5)(x + 3)(x + 1.5)
roots -3 and -1.5 are not fixed points of f(x) and so must form a 2-cycle of f(x).
f'(-3) = -2.5 , f'(-1.5) = +0.5
so | f'(-3).f'(-1.5) | > 1, so 2-cycle is repelling.

(c) (i) fixed point is reached exactly after one iteration.
(ii) long term behaviour is a 4-cycle.

Qn.19 (a) (i) Use either
$u = \sqrt{9 + x^2}$ ($\frac{du}{dx} = \frac{x}{\sqrt{9 + x^2}}$)
or $u = 9 + x^2$ ($\frac{du}{dx} = 2x$) and result follows.

(ii) $\int_0^4 dx = \int_3^5 du = 5 - 3 = 2$

(b) (i) integration by parts with *f(x)*=x^2 and *g'(x)*=cos*(2x)* : $I = \frac{1}{2}x^2 \sin 2x - \int x \sin 2x dx$
from which result follows.

(ii) Vol. of revolution =
$\int_0^p p x^2 \sin^2 x dx = \frac{p}{2}\int_0^p x^2(1 - \cos 2x)dx$
$= \frac{p}{2}\int_0^p x^2 dx - \frac{p}{2} I$

$\int_0^p x^2 dx = \frac{p^3}{3}$ and $[I]_0^p = (\frac{p}{2} - 0)$

so volume $= \frac{p^4}{6} - \frac{p^2}{4}$ = 13.77 to 2 d.p.

Qn.20 (a)
30 = 1.19 + 11
19 = 1.11 + 8
11 = 1.8 + 3
8 = 2.3 + 2
3 = 1.2 + 1

This is an equation containing 1 so work backwards:
1=3 - 1.2
=3 – 1(8-2.3) = -1.8+3.3
=-1.8 +3(11 – 1.8) = 3.11 - 4.8
= 3.11 - 4.(19 - 1.11) = -4.19 + 7.11
= -4.19 + 7.(30 - 1.19)
= -11.19 in modulo 30 arithmetic
= 19.19

so inverse of 19 is itself.

(b) 31 is a prime number, 19 is coprime with 30, and 30 exceeds the number of letters in the alphabet.

(c)
28^{19}=19(mod 31)
13^{19}=21(mod 31)
12^{19}= 3(mod 31)
5^{19}= 5(mod 31)

so message is SUCCESS.

There are many 'short-cut' ways to obtain the deciphered values. One suggestion using $k^{16}.k^2=k^{18}$ is:

power =	1	2	4	8	16	18	19
28	-3	9	-12	-11	-3	4	19
13	13	14	10	7	18	4	21
12	12	-11	-3	9	-12	8	3
5	5	- 6	5	-6	5	1	5