MS221 – 2003 Solutions

Qn.1 (a) $t_{-4,5}$ (see p.48 in Handbook)
(b) (x+3, y+2) (c) $(y-2)^2 = 7(x-3)$

Qn.2 (a) curve is $\frac{x^2}{25}+\frac{y^2}{4}=1$
which is an ellipse in standard form.
(b) (i) $e^2 = 1-4/25 = 21/25$ so $e = \frac{\sqrt{21}}{5}$
(ii) Foci are $(\pm\sqrt{21}, 0)$
(iii) directrices are $x = \pm 25/\sqrt{21}$

Qn.3. $\cos 2x = 1 - 2\sin^2 x$, so
$\sin^2(x/2) = (1 - \cos x)/2$

Put $x = \frac{\pi}{4}$ and multiply top & bottom by 2:

$$\sin\left(\frac{\pi}{8}\right) = \sqrt{\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)} = \sqrt{\frac{1}{4}\left(2-\sqrt{2}\right)} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

Qn.4 (a) $u_2 = 1 + 6(3) = 19$
(b) Solve $r^2-r-6=0$, i.e. $(r-3)(r+2)=0$ to give r=3 and r = -2 as roots.
General solution is $u_n = A(3^n) + B(-2)^n$
Use u_0 and u_1 : A+B = 3
3A - 2B = 1
Solve to give A = 7/5, B = 8/5, so closed form
is $u_n = \frac{7}{5}3^n + \frac{8}{5}(-2)^n$ (n=0,1,2,..)
(c) $u_9 = \frac{7}{5}3^9 + \frac{8}{5}(-2)^9$ which = 26,737

Qn.5 (a) f(x)=x simplifies to $2x^2-3x+1 = 0$, i.e. $(2x - 1)(x - 1) = 0$ so fixpoints are at x=0.5 and x=1.
(b) $f'(x) = -2x + 2.5$, so $|f'(x)| > 1$ for x=0.5 (repelling) and <1 for x=1 (attracting).
(c) see p. 57 of Handbook for graphical criterion : required interval is (0.5,1.25)

Qn.6 (a) $\begin{pmatrix} 7 & -2 \\ 6 & -3 \end{pmatrix}$
(b) (i) |det| = |-21 + 12| = 9 so scaling factor is 9 (see p.60 in Handbook) , and area of triangle = 4.5.

(c) Reqd. matrix is inverse = $\frac{1}{9}\begin{pmatrix} 3 & -2 \\ 6 & -7 \end{pmatrix}$.

Qn.7 (a) eigenlines are $\begin{pmatrix} 7-5 & 5 \\ -4 & -5-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$
and $\begin{pmatrix} 7+3 & 5 \\ -4 & -5+3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

that is 2x = -5y and 2x = -y.

Qn.8 (a) $x_2=(-2,-4)$, $x_3=(4,8)$
(b) $x_2=(2,1)$, $x_3=(8,7)$ – n.b. exact values not required in part (b)
Sequence is tending towards the dominant eigenline y = ¾x (that is the one corresponding to the largest eigenvalue in absolute value) with successive points on either side of it.
[**A** = $\begin{pmatrix} 6 & -4 \\ 6 & -5 \end{pmatrix}$ although you do not require to establish this.]

Qn.9 (a) $f'(x) = 3x^2 \arccos x - x^3/\sqrt{1-x^2}$
using Product Rule
(b) $g'(x) = 5e^{5x}\cos(e^{5x})$ using Composite Rule

Qn.10 (a)
$$2\sqrt{x}.\ln x - \int \frac{2\sqrt{x}}{x}dx = 2\sqrt{x}.\ln x - 4\sqrt{x} + c$$
(b) $du = \frac{1}{3}e^{x/3}dx$ so Integral =
$$\int \frac{3du}{u^2} = -\frac{3}{u} + c = -\frac{3}{e^{x/3}+1} + c$$

Qn.11 (2003 students)
(a) $\frac{d}{dx}(\tan^3 x) = 3\tan^2 x.\sec^2 x$
using Composite Rule, hence result.
(b) volume =
$$\pi\int_0^{\pi/3}(\sec x \tan x)^2 dx = \frac{\pi}{3}\left[\tan^3 x\right]_0^{\pi/3} = \pi\sqrt{3}$$

Qn.11 (2002 students)
(a) $\sec^2 y = 2x dx$
integrate : $\tan y = x^2 + c$
so $y = \arctan(x^2+c)$ is general soln.
(b) Put x = 0 in line 2
$c = \tan(-\frac{\pi}{4}) = -1$ so particular. soln is
$y = \arctan(x^2-1)$

Qn.12 (a) $1 - 5x + 25x^2 - 125x^3$
(b) $x^2 - 5x^3 + 25x^4 - 125x^5$
(c) $-0.2 < x < 0.2$

Qn.13 (a) $\sqrt{58}$, 7 – 3i, 14, 58
(b) $x^2 - 14x + 58$

Qn.14 (a)

	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

(b) 6 – since its row and column are identical with the row and column headers.
(c) $2^{-1}=8$, $4^{-1}=4$, $6^{-1}=6$, $8^{-1}=2$

Qn.15 27=2.10 + 7
10=1. 7 + 3
7=2. 3 + 1

this is an equation for 1 so work backwards:

1= 7 - 2.3
= 7 - 2(10-1.7) = -2.10+3.7
=-2.10 +3(27 - 2.10)
=-8.10 +3.27
=19.10 + 0 in mod 27 arithmetic, so required inverse is 19.

Qn.16 (a) $b(n) \wedge d(n)$

(b) any odd multiple of 12, e.g. 12, 36

(c) (i) $c(n) \Rightarrow (a(n) \wedge b(n))$

(ii) If n is divisible by 24 then n is divisible by 6 and is also divisible by 12.

Qn.17 Use p. 49 in Handbook throughout qn.

(a) Asymptotes are $y = \pm \sqrt{3}/2\, x$

(b) $\theta = \frac{1}{2}\arctan\left(\frac{14\sqrt{3}}{5-(-9)}\right) = \frac{\pi}{6}$

$$A' = 5\left(\frac{3}{4}\right) + 14\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - 9\left(\frac{1}{4}\right) = 12$$

$$C' = 5\left(\frac{1}{4}\right) - 14\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - 9\left(\frac{3}{4}\right) = -16$$

F' = -432

so L is $12x^2 - 16y^2 - 432 = 0$ which is equivalent to K on division by 432.

(c) sketch should show L identical to K but with its axes rotated anticlockwise by $\pi/6$

(d) asymptote with $\phi = \arctan \sqrt{3}/2$ has gradient

$$\arctan\left(\frac{\sqrt{3}/2 + 1/\sqrt{3}}{1-\left(\sqrt{3}/2\right)\left(1/\sqrt{3}\right)}\right) = \arctan\frac{5}{\sqrt{3}}$$

asymptote with $\phi = \arctan\left(-\sqrt{3}/2\right)$ has gradient

$$\arctan\left(\frac{-\sqrt{3}/2 + 1/\sqrt{3}}{1+\left(\sqrt{3}/2\right)\left(1/\sqrt{3}\right)}\right) = \arctan\left(-1/3\sqrt{3}\right)$$

so eqns. are $y = 5/\sqrt{3}\, x$ and $y = -1/3\sqrt{3}\, x$

Qn.18

(a) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 6 \end{pmatrix}$

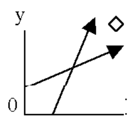
rotation shear scaling

(b) $\begin{pmatrix} 0 & 2 \\ -1 & 6 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0 & 2 \\ -1 & 6 \end{pmatrix}\begin{pmatrix} 8 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

(c) (4,4), (3,4) and (4,3)

(d)



diamond is at (4,4), arrowheads are at (3,4) and (4,3) drawn from (0,1) and (1,0) respectively.

(e) A rotation of $\pi/2$ about (2,2).

Qn.19 (a) (i) Solve g(x) - h(x) = 0

(ii) Write f(x) = xlnx – 1 so f(1) = -1, f(e)=e-1 which >0. f(x) is continuous and so has a solution in (1,e).

(iii) f'(x) = lnx + 1 hence x - f((x)/ f'(x) =

$$x - \frac{x\ln x - 1}{1+\ln x} = \frac{1+x}{1+\ln x}$$

from which the N-Raphson formula follows.

(iv) 1.763255 and 1.763223

(v) x co-ordinate is 1.76 to 2 d.p., y co-ord is h(1.76) = 0.57 to 2 d.p. so curves meet at (1.76, 0.57)

(b) (i) $f(x) = 4\left(\frac{\pi}{4} - x\right)\cos x$

cosx is +ve in both ranges so sign of f(x) is the sign of the quantity in parentheses.

(ii) Integrating by parts gives

$$\sin x(\pi - 4x) + 4\int \sin x\,dx = [\sin x(\pi - 4x) - 4\cos x]$$

which evaluates to –4, $\frac{-4}{\sqrt{2}}$ and $-\pi$ at $0, \frac{\pi}{4}, \frac{\pi}{2}$

so, making all areas positive, total area is

$$2f\left(\frac{\pi}{4}\right) - f(0) - f\left(\frac{\pi}{2}\right) = -4\sqrt{2} + 4 + \pi$$

= 1.485 to 3 d.p.

Qn.20 (a) $<1, 2k\pi/7>$ k=0,1,..6

(b) $<1,0>$ is self-inverse, other inverses are paired: $\{<1,\frac{2\pi}{7}>, <1,\frac{-2\pi}{7}>\}$,

$\{<1,\frac{4\pi}{7}>, <1,\frac{-4\pi}{7}>\}$, $\{<1,\frac{6\pi}{7}>, <1,\frac{-6\pi}{7}>\}$

(c) $p^7 = q^7 = 1$, so $p^7q^7 = (pq)^7 = 1$.

(d) When complex numbers are multiplied, arguments are added, and modulo 7 arithmetic applied, hence H is a <u>closed</u> set under complex multiplication.

1 is an <u>identity element</u>

every element has an <u>inverse</u> by part(b), and multiplication of complex numbers in general is <u>associative</u>, hence H is a group.

(e) H is isomorphic to the group formed by the addition table of {0,1,2,3,4,5,6} modulo 7. (see Handbook p.87) This is because multiplying complex numbers means that their arguments must be added.