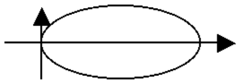
MS221 – 2002 Solutions

Qn.1 (a) (i) $t_{7,-3}$ (ii) $t_{-6,3}$
(b) (i) $t_{7,0}$ (ii) (8,2)

Qn.2 (a) curve is $\frac{(x-3)^2}{9}+\frac{y^2}{1}=1$
which is an ellipse in standard form.
(b) centre is at (3,0), axis length are 6 and 2.
(c) x-axis : Solve vs. $y=0$ to give $x=0$ or 6
y-axis : Solve vs. $x=0$ to give $y=0$



Qn.3. cos2x = 1 - 2sin^{2}x, so

$$\frac{\sqrt{3}}{2}=1-2\sin^2(\frac{\pi}{12}) \text{ leading to}$$

$$\sin(\frac{\pi}{12})=\sqrt{\frac{1}{2}\left(1-\frac{\sqrt{3}}{2}\right)}$$

Qn.4 (a) u_2=6(2) – 9 = 3
(b) Solve r^2-6r+9=0 to give r=3 as a double root.
General solution is $u_n=(A+Bn)(3^n)$
Use u_0 and u_1 : A = 1
3A + 3B = 2
Solve to give A = 1, B = -1/3, so closed form is $u_n=\left(1-\frac{1}{3}n\right)(3^n)$ (n=0,1,2,..)

Qn.5 (a) $(3-2x)^4$ =
$81+4.27(-2x)+6.9.(-2x)^2+4.3(-2x)^3+(-2x)^4$
$= 81 - 216x + 216x^2 - 96x^3 + 16x^4$

Qn.6 (a) (0,1] and $(-\infty,\infty)$ (Note that since $\frac{\pi}{2}$ is in the domain, the image set includes 1).
(b) (0,1] is within the domain $(0,\infty)$ of g so g o f can be formed.
(c) $g\circ f:(0,\pi)\to(-\infty,0]$
$x\mapsto \ln(\sin x)$

Qn.7 (a) eigenlines are $\begin{pmatrix}5-4 & -5\\ 2 & -6-4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$=0
and $\begin{pmatrix}5-(-5) & -5\\ 2 & -6-(-5)\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$=0

that is x = 5y and 2x = y.
(b) points remain on the line x=5y, moving progressively further out by a multiple of 4 for each iteration, i.e. (20,4), (80,16) etc.

Qn.8
(a) $\begin{pmatrix}\frac{3}{5} & -\frac{4}{5}\\ \frac{4}{5} & \frac{3}{5}\end{pmatrix}\begin{pmatrix}5 & 0\\ 0 & 10\end{pmatrix}=\begin{pmatrix}3 & -8\\ 4 & 6\end{pmatrix}$
(b) (0,0), (3,4), (8,6), (5,10)
(c) f : scaling (by factors (5,10)),
g : rotation (by $\tan^{-1}$ (4/3) about 0)

Qn.9 (a) $f'(x)=\frac{e^{7x}(7x-2)}{x^3}$

(b) $g'(t)=\frac{1/2\sqrt{t}}{\sqrt{1-t}}=\frac{1}{2\sqrt{t(1-t)}}$

Qn.10 (a)

$$\frac{1}{4}x\sin(4x)-\frac{1}{4}\int\sin(4x)dx=\frac{1}{4}x\sin(4x)+\frac{1}{16}\cos(4x)+c$$

(b) u = ln x ; du = (1/x) dx
Integral = $\int u\,du=\frac{1}{2}(\ln x)^2+c$

Qn.11(a) $f(x)=2x+\frac{2x^3}{3}+\frac{2x^5}{5}+\frac{2x^7}{7}$

(b) $f'(x)=2+2x^2$

Qn.12 (a) $e^{-3y}dy=dx$
Integrate $-\frac{1}{3}e^{-3y}=x+k$
Solution : $e^{-3y}=-3x+c$ that is
y = -(ln(-3x+c))/3

(b) At x=0,y=1 so –3+c = 1, i.e. c=4
so particular soln. is y=-(ln(4-3x))/3

Qn.13 (a) $<1,3\pi/4>$
(b) $<1,\pi/4>,<1,11\pi/12>,<1,19\pi/12>$

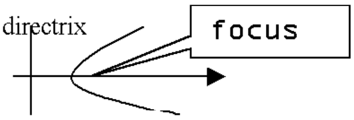
Qn.14 (a) 11^2 = 121 = 2 (mod 17)
(b) $11^7 = 11^4 . 11^2 . 11$ = 4.2.11 (mod 17)
= 88 (mod 17) = 3

Qn.15 (a) $d*b=c$,$f*d=b$
(b) c beca$_6$se c row in table is the same as the column headings.
(c) $a^{-1}=f$, $d^{-1}=b$
(d) ($\mathbf{Z}_6$, $+_6$) since there are just two self-inverse elements , namely c and e.

Qn.16 (a) $a(n)\wedge b(n)$
(b) any odd multiple of 42, e.g. 42
(c) (i) $d(n)\Rightarrow(a(n)\wedge c(n))$
(ii) If n is divisible by 84 then n is divisible by 7 and is also divisible by 42.

Qn.17 (a) New eqn. is $(y-1)^2=12(x+2)$
which simplifies to the eqn. Q.
(b) focus : (1,1) directrix : $x=-5$,
axis of symmetry : $y=1$.
(c) x-axis : Solve Q vs. $y=0$ to give $x=-23/12$
y-axis : Solve Q vs. $x=0$ to give

$$y = 1 \pm \sqrt{1+23} = 1 \pm \sqrt{24}$$

(d)


(e) $t=(y-7)/6$ so $x=((y-7)^2)/12 + (y-7)+1$
multiply by 12 :
$12x = y^2-14y+49 + (12y-84)+12$
i.e. $y^2 - 2y - 12x + 23=0$

Qn.18 (2002 students) (a) Find where the curve meets the line $y = x$.
(b) Solve $-x^2 + 2x + 1 = x$, i.e.
$x^2 - x - 1 = 0$. Solutions are

$x = \frac{1}{2}(1 \pm \sqrt{5}) = 1.618$ or -0.618

(c) gradient of $-x^2 + 2x + 1$ is $-2x + 2$ which exceeds 1 in absolute value at both 1.618 and -0.618 so both fixpoints are repelling.
(d) (i) 2, 1, 0
(ii) values 1 and 2 show that there is a two-cycle (1,2) and 0 shows it is super-attracting.
(e) (i) 1.06, 2.00
(ii) a two-cycle between 1 and 2.

Qn.19 (a)

$$\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u = \frac{1}{2}\ln(1+x^2)+c$$

(b) Using integration by parts, integral =

$$x \arctan x - \int \frac{x\,dx}{1+x^2} = x \arctan x - \frac{1}{2}\ln(1+x^2)+c$$

(c) in above expression put x=1 :

$$\frac{\pi}{4} - \frac{\pi}{4} - \frac{1}{2}\ln(2) + c$$

so c = ½ ln(2) and solution is

$x \arctan x - \frac{1}{2}\ln(1+x^2) + \frac{1}{2}\ln(2)$ that is

$$y = x \arctan x + \frac{1}{2}\ln\left(\frac{2}{1+x^2}\right)$$

(d) $y(0) = \frac{1}{2}\ln(2)$
$y'(x) = \arctan x$ $\quad y'(0) = 0$
$y''(x) = \frac{1}{1+x^2}$ $\quad y''(0) = 1$
Taylor series is $\frac{1}{2}\ln(2) + \frac{1}{2}x^2$

Qn.20 (a)(i)

	1	4	13	16
1	1	4	13	16
4	4	16	1	13
13	13	1	16	4
16	16	13	4	1

(ii) it satisfies closure;
there exists an identity element 1
every element has an inverse, viz:
(1,1) (4,13) (13,4) (16,16)
the operation is associative since multiplication in general is associative.

(b) (i) I : identity
H : half-turn
P : reflection about $y = x$
Q : reflection about $y = -x$

group table is:

I	H	P	Q
H	I	Q	P
P	Q	I	H
Q	P	H	I

(ii) $I^{-1}=I$, $H^{-1}=H$, $P^{-1}=P$, $Q^{-1}=Q$

(c) no – one has two identity elements on the diagonal, the other has four.