MS221 – 2001 Solutions

Qn.1 (a) (i) $t_{-3,2}$ (ii) $t_{4,1}$ (b) $t_{5,4}$

Qn.2 (a) curve is $y^2=8x$, and so is a parabola with apex at (0,0) and axis horizontal.
(b) one focus at (2,0), one directrix at x=-2, eccentricity = 1

Qn.3 sinx.cosx = ½sin2x, so 4sinx.cosx = 2sin2x which ranges from –2 to 2, so image set is [2,6]. *** **n.b. qn no longer in syllabus**

Qn.4 (2001) (a) $u_2=8-18=-10$
(b) Solve $r^2-4r+3=0$ to give r=3 and r=1.
General solution is $u_n = A(3^n)+B(1^n)$
Use u_0 and u_1: A + B = 6
3A + B = 2
Solve to give A = -2, B = 8, so closed form is
$u_n = 8 - 2(3^n)$ (n=0,1,2,..)

Qn.4 (2000) (a) $w_1 \ldots w_4 = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

(b) $u_n = \frac{1}{n} - \frac{1}{n-1} = \frac{1}{n(n+1)}$

(c)
$w_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \ldots = 1 - \frac{1}{(n+1)} = \frac{n}{(n+1)}$

Qn.5 (a) decreasing
(b) Suppose $0 < x_1 < x_2$
$(1/x_1) - (1/x_2) = (x_2 - x_1)/ x_1.x_2$
numerator and denominator are both >0, and so $f(x_1) > f(x_2)$.
(c) $(0, \infty)$

Qn.6 (a) $(1-3x)^5=$
$1+5(-3x)+10(-3x)^2+10(-3x)^3+5(-3x)^4+(-3x)^5$
$=1-15x+90x^2-270x^3+405x^4-343x^5$
(b) Put x=0.001
first 3 terms=1–0.015+0.00009
= 0.985090 to 6 d.p.
$|270x^3|= 0.00000027$, so this and subsequent terms are too small to count.

Qn.7 (a) for fixed points solve $kx(1-x)=x$, i.e.
$kx^2-(k-1)x = x(kx-(k-1))=0$
Soln. is $x=0$ or $x=(k-1)/k = 1-1/k$
(b) k=2.5, 1-1/k = 1-(2/5) = 0.6, so fixed points are 0 and 0.6.
(c) $g'(x) = 2.5-5x$ $\quad |g'(0)| = 2.5$ $\quad |g'(0.6)| = 0.5$
and so 0 is repelling and 0.6 is attracting.

Qn.8 (a) $Q^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

$M^5 = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -32 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -\frac{64}{3} & \frac{32}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} -21 & 11 \\ 22 & -10 \end{pmatrix}$

Qn.9 (a) $f'(x) = \frac{\sec x}{x} + \sec x \tan x \ln x$
using product rule

[or $f'(x) = \sec^2 x(\frac{\cos x}{x} + \sin x \ln x)$
using quotient rule

(b) $g'(t) = \frac{e^t}{1+e^{2t}}$ by chain rule

Qn.10 (a)
$\frac{1}{3}x^3\ln(5x) - \frac{1}{3}\int x^2 dx = \frac{1}{3}x^3\ln(5x) - \frac{1}{9}x^3 + c$

(b) u = 5 – sin x ; du = -cos x dx
Integral = $\int -e^u du = -\exp(5-\sin x) + c$

Qn.11(a) $f(x) = -4x - \frac{16x^2}{2} - \frac{64x^3}{3} - \frac{256x^4}{4}$
$= -4x - 8x^2 - \frac{64x^3}{3} - 64x^4$

(b) $-\frac{1}{4} < x < \frac{1}{4}$
(c) -4-16x

Qn.12 (a) $\frac{1}{y^2}dy - \sin x dx$

Integrate $\frac{1}{y} = \cos x + c$

Solution : $y = 1/(\cos x + c)$

(b) At x=0,y=1/3 so 3 = 1+c
=> c= 2 and part. soln is $y = 1/(\cos x + 2)$

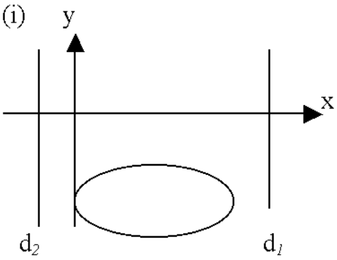
Qn.13 (a) 13, 5-12i, 10, 169
(b) $(x-(5+12i))(x-(5-12i))$
$= x^2-10x+169$

Qn.14 (a) 5 (b) any of 0,2,4,6,7,8,10,12

Qn.15 (a) $e*b=a$, $c*e=f$
(b) d because d row in table is the same as the column headings.
(c) $c^{-1}=a$, $f^{-1}=b$
(d) $+_6$ since there are just two self inverse elements , namely d and e.

Qn.16 26=1.15 + 11
15=1.11 + 4
11=2.4 + 3
4=1.3 + 1
this is an equation for 1 so work backwards:
1=4 - 1.3
=4 - 1(11-2.4) = -1.11+3.4
=-1.11 +3(15 - 1.11)
= 3.15-4.11 = 3.15-4(26-1.15)
= 7.15 in mod 26 arith.
so required inverse is 7.

Qn.17 (a) $\frac{(x-2)^2}{4}+(y+3)^2=1$
(b) Substitute for x and y in eqn. of C:
$$\frac{(2\cos\theta+2-2)^2}{4}+(\sin\theta-3+3)^2$$
$$=\cos^2\theta+\sin^2\theta=1$$
(c) $e=\sqrt{1-b^2/a^2}=\sqrt{1-1/4}=\sqrt{3}/2$
Foci = $(\pm\sqrt{3},0)$ Directrices: $x=\pm\frac{4}{3}\sqrt{3}$
Major axis length =4, minor =2
(d) Foci = $(2\pm\sqrt{3},-3)$
Directrices are $x=2\pm\frac{4}{3}\sqrt{3}$
(e) (i)



(foci not shown)

(ii) C touches y-axis at (0,-3)
(iii) axes of symmetry are x = 2 and y =-3

Qn.18 (a) (i) (x,y) is transformed into (y,-x) so transformation is clockwise rotation through an angle of $\pi/2$.
(ii) Because eigenvalues are solutions of $k^2+1=0$ which are imaginary.
(iii) Four, since 4 rt. angles = a complete revolution.

(b) (i) see p.57 in Handbook: reflection in line making angle of $\alpha=\tan^{-1}(\frac{1}{2})$ with x axis has matrix representation $\begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix}$
$$\sin 2\alpha=2\sin\alpha\cos\alpha=2.\tfrac{1}{\sqrt5}.\tfrac{2}{\sqrt5}=4/5$$
and $\cos 2\alpha=\sqrt{1-\sin^2 2\alpha}=3/5$. Hence R is the required matrix.
(ii) if n is odd, the result is equivalent to a single application of R, if n is even it is an identity transformation.

(c) (i) T= $\begin{pmatrix}\frac35 & \frac45\\ \frac45 & -\frac35\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}-\frac45 & \frac35\\ \frac35 & \frac45\end{pmatrix}$
(ii) (1,0), (0,1) are transformed to (-4/5,3/5) and (3/5,4/5) respectively and (1,3) is transformed into itself.
T is a reflection in the line y=3x.

Qn.19
(a) $\pi/8=0.393$ to 3 d.p.
(b) $\frac{d}{dx}\tan x=\sec^2 x$ which = 1 at x=0 so tangent approximation is $p(x)=x$.
(c) $\pi^2/32=0.308$ to 3 d.p.
(d) Integral = -ln(cos x) between limits 0 and $\pi/4$ which = $\ln(\sqrt{2})=0.347$ to 3 d.p.

Qn.20 (a) (i) B (ii) Consider e.g. p=9, q=1
(iii) Proof by counter-example
(b) Proof by induction:
Differentiate given result:
$$f^{(n+1)}(x)=2^{n-1}e^{2x}[2(n+2x)+2]$$
$$=2^n e^{2x}[(n+1)+2x]$$
which is the same as $f^{(n)}$ with $n+1$ replacing n.
Also $f'(x)=e^{2x}[2x+1]$ by calculus, which is the same as the given result with n=1
Hence by induction the result is true for all n.