MS221/P

The Open University

Second Level Course Examination 2000
Exploring Mathematics

Tuesday, 17 October, 2000 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can. You should attempt not more than **TWO** questions in Part II. Your answers to each part should be written in the answer books provided.

72% of the available marks are assigned to Part I and 28% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

Graph paper is available from the invigilator, if you feel it would assist you in answering questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Put your answer books together, with your signed desk record on top. Fix them all together with the fastener provided.

PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part I carries 72% of the available examination marks. Each question carries an indication of the number of marks that are allocated to it.*

(iii) *You should record your answers to each question in the answer book(s) provided. You are strongly advised to show all your working, including any rough working.*

Question 1 – 3 marks

Express the recurring decimal 0.5764 5764 5764... as a fraction. (You do not need to give your answer in its simplest form.) [3]

Question 2 – 6 marks

The equation

$$x^2 + 4y^2 - 4x - 12 = 0$$

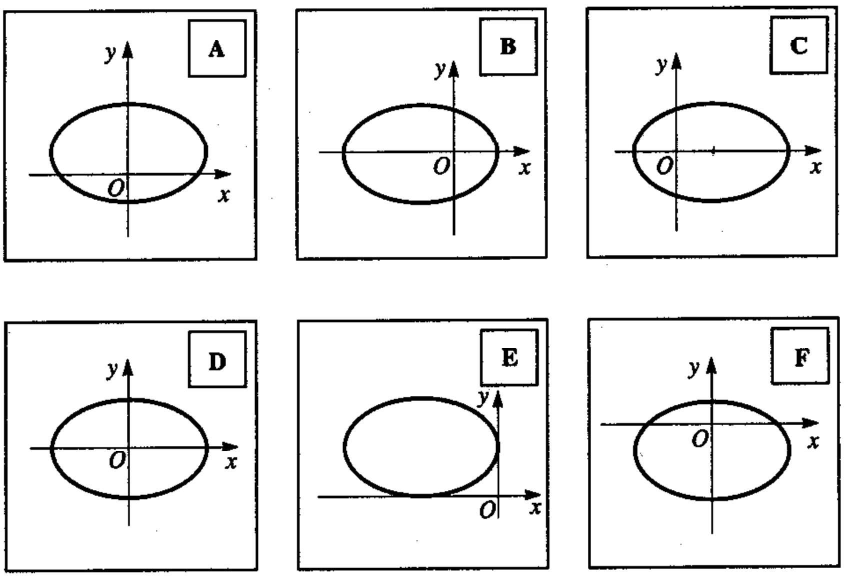
represents a conic.

(a) Show that this conic is an ellipse. [1]

(b) (i) Find where the ellipse crosses the x-axis.

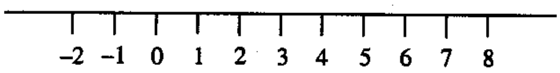
(ii) Show that replacing y by $-y$ leaves the equation unchanged. [3]

(c) One of the following diagrams shows a sketch of the ellipse. Write down which diagram this is and explain your choice, briefly. [2]



Question 3 – 4 marks

This question concerns transformations of the number line.



(a) (i) Write down a translation that sends the point 7 to the point 1.

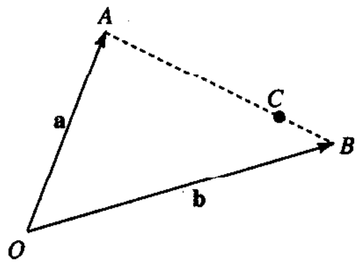
(ii) Write down a reflection that sends the point 7 to the point 1. [2]

(b) Solve the following decomposition problem for the unknown reflection, $\mathbf{ref}_p$:

$$\mathbf{ref}_3 \circ \mathbf{ref}_p = \mathbf{trans}_{10}.$$ [2]

Question 4 – 5 marks

In the diagram below, the directed line segments OA and OB represent the vectors **a** and **b** respectively, and C is a point on AB such that $AC = 3CB$.



(a) Find, in terms of **a** and **b**, the vector that represents the directed line segment

(i) AB;

(ii) AC. [2]

(b) Hence find the vector **c**, that represents the directed line segment OC, in terms of **a** and **b**. [1]

(c) In the case where $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ 8 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 8 \\ 3 \\ 0 \end{pmatrix}$, find the vector **c**. [2]

Question 5 – 3 marks

Consider the function defined by

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} 2x+1 & (x \le 0), \\ x-1 & (x > 0). \end{cases}$$

(a) State whether or not this function is one-one. [1]

(b) Justify your answer. [2]

Question 6 – 5 marks

(a) Use the Binomial Theorem to obtain the expansion of $(1+k)^7$ as a sum of powers of k. [2]

(b) (i) Use the expansion that you found in part (a) to evaluate 0.995^7 correct to 3 decimal places.

(ii) Explain, briefly, the reasoning behind your method. [3]

Question 7 – 5 marks

Consider the function

$$f(x) = x^2 - 4x + 6 \quad (x \in \mathbb{R}).$$

(a) Show that 2 and 3 are fixpoints of f. [1]

(b) Draw a sketch of the graph of f, together with the line $y = x$, showing the fixpoints. [2]

(c) Explain briefly, with reference to your sketch, whether each fixpoint is attracting or repelling. [2]

Question 8 – 5 marks

A rotation, through an angle θ about the origin, maps the x-axis onto the line $7y = 4x$.

(a) Find $\sin\theta$ and $\cos\theta$, leaving your answers in surd form. [2]

(b) Write down the matrix of this transformation. [1]

(c) Find the inverse of the matrix that you wrote down in part (b), and explain what transformation it represents. [2]

Question 9 – 4 marks

Differentiate the following functions. In each case, state which of the principal rules of calculus you are using.

(a) $f(x) = x \arcsin x \quad (-1 < x < 1)$ [2]

(b) $g(t) = \cos(\ln t) \quad (t > 0)$ [2]

Question 10 – 5 marks

(a) Using integration by parts, find the indefinite integral

$$\int (x+2)\exp(-x)\,dx.$$ [3]

(b) Using the substitution $u = 1 + \tan x$, or otherwise, find the indefinite integral

$$\int \frac{\sec^2 x}{1 + \tan x}\,dx \quad \left(-\tfrac{1}{4}\pi < x < \tfrac{1}{2}\pi\right).$$ [2]

Question 11 – 4 marks

(a) Use a result from the Handbook to write down the first four non-zero terms of the Taylor series about $x = 0$ for the function

$$f(x) = \frac{1}{1+3x}.$$ [2]

(b) For what maximum range of values of x does this Taylor series converge to the given function $f(x)$? [1]

(c) Without differentiating $f(x)$ directly, write down the first three non-zero terms of the Taylor series about $x = 0$ for the function $f'(x)$. [1]

Question 12 – 5 marks

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = y^{1/2} \cos x \quad (y > 0).$$ [3]

(b) Find the particular solution of this differential equation that satisfies the initial condition $y = 1$ when $x = 0$. [2]

Question 13 – 5 marks

Let $z = \langle 16, \frac{1}{3}\pi \rangle$.

(a) Find two complex numbers w, in polar form, such that $w^2 = z$. [3]

(b) Express the two numbers that you found in part (a) in the form $x + iy$. [2]

Question 14 – 4 marks

In this question, you should find the required remainders by repeated squaring and congruence properties. If you use any other method, you will not receive any marks.

The remainder when 40 is divided by 17 is 6.

(a) Find the remainder when 40^2 is divided by 17. [1]

(b) Find the remainder when 40^7 is divided by 17. [3]

Question 15 – 4 marks

Consider the group $(G, *)$ whose Cayley table is given below.

$*$	a	b	c	d
a	c	a	d	b
b	a	b	c	d
c	d	c	b	a
d	b	d	a	c

(a) Which element is the identity element of $(G, *)$? [1]

(b) Write down the inverse of (i) a; (ii) c. [1]

(c) To which of the groups listed in the Handbook is $(G, *)$ isomorphic? Justify your answer briefly. [2]

Question 16 – 5 marks

The variable propositions $a(n)$, $b(n)$, $c(n)$ and $d(n)$, where $n \in \mathbb{N}$, have the meanings given below.

$a(n)$ means: n is divisible by 5
$b(n)$ means: n is divisible by 6
$c(n)$ means: n is divisible by 12
$d(n)$ means: n is divisible by 90

(a) Using a combination of the propositions above, give a condition that is necessary and sufficient for n to be divisible by 60. Write your answer in symbols. [2]

Consider the proposition (A) given below.

(A) $\quad (a(n) \wedge b(n)) \Rightarrow d(n)$.

(b) Give an example of a number n in $\mathbb{N}$ for which proposition (A) is false. [1]

(c) Write down the converse of proposition (A)

(i) in symbols;

(ii) in English. [2]

PART II

Instructions

(i) *You should attempt not more than* **TWO** *questions from this part of the examination.*

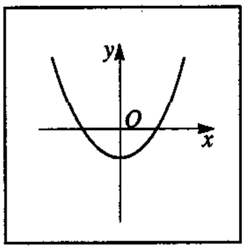
(ii) *Each question in this part carries 14% of the marks.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

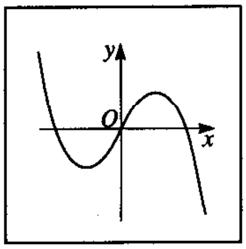
(iv) *Show all your working.*

Question 17

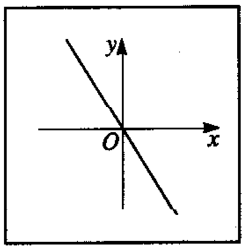
This question concerns symmetry properties of the figures below. (The axes are not part of the figures.)



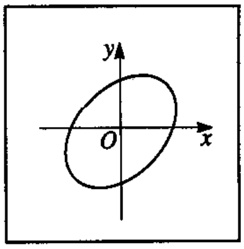
A The parabola $y = x^2 - 1$



B The cubic $y = 4x - x^3$



C The line $y = -2x$



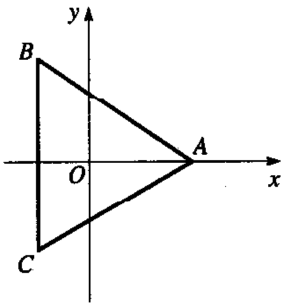
D An ellipse

(a) (i) Explain briefly, in terms of transformations, what is meant by a line of symmetry of a figure. [1]

(ii) For each of the figures above say how many lines of symmetry the figure has and identify those lines. [4]

(b) Which of the figures above have rotations other than the identity that leave the figure setwise invariant? In each case specify the centre of the rotation and the angle(s) of rotation. [3]

The figure shows an equilateral triangle, in the plane, with its centre at the origin. Two isometries of the plane that leave this triangle setwise invariant are q_0 (reflection in the x-axis) and $r_{2\pi/3}$ (rotation through $2\pi/3$ anticlockwise about the origin).



(c) (i) Using the same standard notation list all the isometries of the plane that leave the triangle setwise invariant, and give a brief description of each one. [4]

(ii) Copy and complete the following table for compositions of the isometries q_0 and $r_{2\pi/3}$.

	q_0	$r_{2\pi/3}$
q_0		
$r_{2\pi/3}$		

[2]

Question 18

Let $\mathbf{M} = \begin{pmatrix} 4 & -6 \\ 1 & -1 \end{pmatrix}$.

(a) Show that the eigenvalues of **M** are 1 and 2, and find the corresponding eigenlines. [4]

(b) Express **M** in the form $\mathbf{QDQ}^{-1}$, where **D** is the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. [3]

(c) Use your results to find $\mathbf{M}^{10}$ as a single matrix. [4]

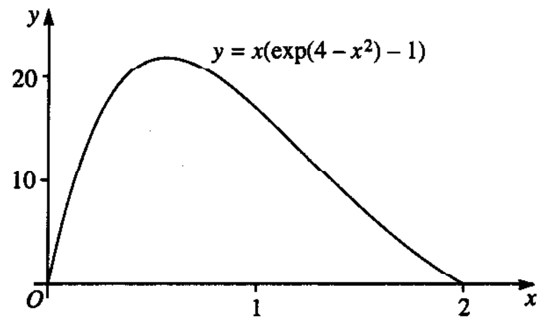
(d) If the matrix **M** were used in the iterative process $\mathbf{x}_{n+1} = \mathbf{M}\mathbf{x}_n$ $(n = 0, 1, 2, \ldots)$, what would happen to the vector $\mathbf{x}_n$ in the long run if

(i) $\mathbf{x}_0 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$;

(ii) $\mathbf{x}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$? [3]

Question 19

The figure below shows part of the graph of the function $y = x\left(\exp(4 - x^2) - 1\right)$.



(a) Show that the derivative of the function is

$$g(x) = \frac{dy}{dx} = (1 - 2x^2)\exp(4 - x^2) - 1.$$ [2]

(b) (i) Show that there is a root of the equation $g(x) = 0$ within the interval $[0, 1]$. [1]

(ii) Derive the Newton-Raphson formula required to find this root. [4]

(c) Mathcad was used to apply the Newton-Raphson method to $g(x) = 0$, with starting value $x_0 = 0.5$. The results obtained were $x_1 = 0.6906$, $x_2 = 0.6965$ and $x_3 = 0.6965$. Use this information to calculate, to one decimal place, the maximum value of $y = x\left(\exp(4 - x^2) - 1\right)$ within the interval $[0, 2]$.
(You may assume that there is only one root of $g(x) = 0$ within $[0, 2]$.) [2]

(d) Find, to one decimal place, the area enclosed by this graph and the x-axis. [4]

(e) You should have found that your answer to part (c) was within 10% of your answer to part (d). With reference to the graph, explain why these values might be expected to be relatively close. [1]

Question 20

(a) (i) Give an example of a non-zero number x in $\mathbb{Z}_{27}$ that does not have a multiplicative inverse. [1]

(ii) What property must the number x have if it has a multiplicative inverse in $\mathbb{Z}_{27}$? [1]

(b) Use Euclid's algorithm to find the multiplicative inverse of 8 in $\mathbb{Z}_{27}$. [7]

(c) A message, using the correspondence shown in the table below, is enciphered using the function

$$f(m) = 8 \times_{27} m.$$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z

The cipher text is

25, 10, 9, 13, 13.

What is the message? [5]

[END OF QUESTION PAPER]