MZS371/C

The Open University

Third Level Course Examination 1998

Computational Mathematics

Tuesday, 13 October, 1998 7.30 pm – 10.30 pm

Time allowed: 3 hours

This paper is divided into **FOUR** parts.

Attempt **SIX** questions of which **NOT MORE THAN TWO** questions should be from any one part of the paper. All questions carry equal weight and are marked out of 15. The final score is out of 90.

Two sheets of graph paper are provided with this question paper for use, if required, in answering questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach all your books together using the paper fastener provided.

Enter the numbers of the questions that you have attempted in the boxes on the front page of the answer book(s).

PART I

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 1

Consider the equation

$$3\sin x = e^{-2x}.$$

(i) Show that there are two roots of this equation in the interval $[0, 4]$. [3]

(ii) Show that the iterative scheme

$$x_{r+1} = g(x_r) = \arcsin\left(\tfrac{1}{3}e^{-2x_r}\right)$$

satisfies the conditions for a contraction mapping on $[0, 1]$ and deduce that, given any $x_0 \in [0, 1]$, the sequence of iterates will converge to a root α of the above equation. [7]

(iii) Starting with $x_0 = 0.5$, compute the root α correct to two decimal places using the scheme in (ii). [2]

(iv) Let $f(x, c) = 3\sin x - ce^{-2x}$, where c is subject to small changes about $c = 1$. Determine the absolute and relative condition numbers for the problem of solving $f(x, c) = 0$ for the root α. Comment on the conditioning of this problem. [3]

Question 2

Consider the system of linear equations $\mathbf{Ax} = \mathbf{b}$ given by

$$\begin{bmatrix} 2 & 4 & -3 \\ 6 & -1 & 2 \\ -8 & 6 & -6.6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}.$$

(i) Obtain the **LU** decomposition of **A** and hence the solution of the system, using four-figure arithmetic throughout. [6]

(ii) Use the full precision of your calculator to compute the residual vector **r** and to carry out one iterative refinement to obtain a revised solution. [5]

(iii) Use your results from (i) and (ii) to comment on the absolute and relative conditioning of the problem. [4]

Question 3

Consider the following system of nonlinear equations:

$$f_1(x_1, x_2) = x_1 x_2 - 3x_2 + 3 = 0$$
$$f_2(x_1, x_2) = x_1^2 + 2x_2^2 - 5 = 0$$

(i) Determine, graphically or otherwise, the number of roots of the system and their approximate locations. [3]

(ii) Determine the Jacobian matrix J for $\mathbf{f} = [f_1, f_2]^T$. Hence, using your results from (i), deduce whether the Newton–Raphson method will converge to each of the roots found in (i), given starting values close to these roots. [3]

(iii) Carry out **one** iteration of the Newton–Raphson method to find an approximate root of the system, starting from $\mathbf{x}_0 = [1, 1.5]^T$. Does this support your conclusion in (ii)? [5]

(iv) What major computational difficulties arise when using the Newton–Raphson method for solving large systems of nonlinear equations? How can these difficulties be overcome? [4]

PART II

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 4

A village on the River Sepik in Papua New Guinea produces wooden carvings: masks, crocodiles and spears. Each carving requires three processes: rough cutting, carving and polishing. For each type of carving, the time required for each process, the sale price in kina, and the maximum number it can sell each day are given in the following table, together with the maximum time available in hours per day for each process.

Process	Mask (minutes)	Crocodile (minutes)	Spear (minutes)	Maximum time (hours per day)
Rough cutting	48	75	84	15
Carving	60	48	72	10
Polishing	102	132	144	19
Selling price (kina)	10	12	14	
Maximum number per day	12	10	10	

The village can also purchase masks and crocodiles from the nearby village of Angoram for 8 kina and 10 kina, respectively. The objective is to maximize daily income. Assume that parts of carvings can be left overnight, if necessary.

(i) Formulate this problem as a linear programming model in standard form. [7]

(ii) Express the problem in canonical form using matrices and vectors. [2]

(iii) Complete **one** iteration of the simplex method for this model, starting from the all-slack solution, and give the solution after one iteration. [6]

Question 5

Consider the following linear programming model:

$$\text{minimize } z = 2x_1 + x_2 + 3x_3$$

subject to

$$4x_1 + 2x_2 - 3x_3 \geq 8$$
$$x_1 + 4x_2 + 2x_3 \leq 15$$
$$2x_1 + x_2 + 3x_3 = 10$$
$$x_1, x_2, x_3 \geq 0$$

CANONICAL

(i) Give the augmented form of this model using matrices, and show that the all-slack solution is infeasible. [4]

(ii) What is the pseudo-objective and the initial pricing vector for the first iteration of Phase I of the revised simplex method, starting from the all-slack solution? [2]

(iii) Complete **one** iteration of the revised simplex method and find a new basic solution. Is this new solution feasible? What is the objective or pseudo-objective function for the second iteration of the revised simplex method? [9]

Question 6

A telecomputer village assembles two types of computers: standard and super-computer. Assembly involves three processes: mother board, cards and power supply, and disk drive and box housing. For each process, the time in minutes for each type of computer and the total time available per day is given in the following table, together with the sale price in pounds for each type of computer.

	Standard (minutes)	Super-computer (minutes)	Total time available (minutes per day)
Mother board	70	48	550
Cards and power supply	30	65	460
Disk drive and box housing	60	68	590
Selling price (£)	1400	2400	

(i) Formulate this problem as an integer programming model, with the objective of maximizing daily sales receipts, given that it can sell all that it makes and that no incomplete computers can be left overnight. [4]

(ii) Solve the problem graphically using the branch-and-bound method to determine the number of each type of computer the village should make each day, using a suitable branching strategy that you should specify. Solve each continuous subproblem graphically. Construct a table of your results along the following lines. [11]

Problem, I_k	Problem solved, C_k	Optimal solution to C_k: x_1	x_2	z	Current bound	Problem to pursue	Stored problems

PART III

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 7

Consider the function

$$f(\mathbf{x}) = 4x_1^3 - 5x_1x_2 + 4x_2^2 + 3x_1 - 7x_2.$$

(i) Perform **one** iteration of the alternating variables method to determine an approximation to a local minimizer of f, starting from $\mathbf{x}^{(0)} = [0, 1]^T$. The line search should be performed analytically and your iterate should be quoted to four decimal places. State the search direction for the next iteration. [6]

(ii) Repeat (i) using the steepest descent method. [8]

(iii) Why might you expect the steepest descent method to converge faster than the alternating variables method? [1]

Question 8

This question is concerned with the minimization of the function

$$f(\mathbf{x}) = x_1^3 - 4x_1x_2 + 10x_2^2 - 5x_1.$$

(i) Complete **one** iteration of the Newton–Raphson method *without* line searches, starting from $\mathbf{x}^{(0)} = [1, 0]^T$, to find an approximate local minimizer of f. [6]

(ii) Complete **one** iteration of the Newton–Raphson method *with* line searches, starting from $\mathbf{x}^{(0)} = [1, 0]^T$, to find an approximate local minimizer of f. Compare this method with the method of (i). [9]

Question 9

Consider the following constrained minimization model:

$$\text{minimize } f(\mathbf{x}) = 3x_1^4 + 2x_1^2x_2 + 4x_2^2 - 5x_1x_3 + 2x_3^2 + 25x_1 + 4x_1x_2 + 16x_2$$
$$\text{subject to } c(\mathbf{x}) = 2x_1 + 2x_2 + x_3 - 3 = 0$$

(i) Write down the simple penalty function, $\phi(\mathbf{x}, \sigma)$, for this model. Indicate how ϕ can be used to solve the given model. [4]

(ii) Show that $\boldsymbol{\alpha} = [1, -1, 3]^T$ is a feasible point and determine the set of all feasible directions at this point. [4]

(iii) Write down the Lagrangian function $L(\mathbf{x}, \mu)$ for this model and determine the corresponding first-order necessary conditions for a constrained local minimizer. [3]

(iv) Show that $\boldsymbol{\alpha} = [1, -1, 3]^T$ is a stationary point of L and find the corresponding value of the Lagrange multiplier, λ. [2]

(v) What else is required to show that $\boldsymbol{\alpha}$ is a constrained local minimizer for f? *(You do not need to do the calculation required to show that* $\boldsymbol{\alpha}$ *is a constrained local minimizer, but you should determine any matrices and vectors that would be required in order to do the calculation.)* [2]

PART IV

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

~~Question 10~~

Consider the definite integral

$$I = \int_0^2 \sqrt{1 + x^5}\, dx$$

which is to be estimated using the following six normalized pseudo-random numbers:

0.755	0.275	0.329	0.923	0.509	0.326

(i) Use the crude Monte Carlo method to estimate the integral I, and determine a 95% confidence interval for your estimate. [7]

(ii) Use antithetic variables and the crude Monte Carlo method to obtain an improved estimate for I, and a corresponding 95% confidence interval. [6]

(iii) Use your results in (ii) to determine the sample size required to estimate I to two decimal places with 95% confidence using antithetic variables. [2]

~~Question 11~~

This question concerns the operation of a small pharmacy, which employs two assistants, Kate and Jane, to serve its customers.

(i) Customers arrive at the pharmacy with inter-arrival times distributed according to a negative exponential distribution with mean 30 seconds. Sample from this distribution, using the following four normalized pseudo-random numbers, to obtain four inter-arrival times, rounded to the nearest second.

0.88	0.31	0.91	0.25

[4]

(ii) Service times for Kate and Jane follow different normal distributions:

	Mean, μ (seconds)	Standard deviation, σ (seconds)
Kate	70	20
Jane	50	10

Use the following six normalized pseudo-random numbers to sample from the normal distributions, rounding your answers to the nearest second.

Kate	0.81	0.23	0.58
Jane	0.12	0.56	0.77

[5]

(iii) Customers arriving at the pharmacy form a single queue and the person at the front of the queue moves to the first assistant to become free. The time taken to move to the service counter is negligible. Use the following inter-arrival and service times (given in seconds) to perform a hand simulation of the operation of the pharmacy, assuming that there are already two customers waiting when the pharmacy opens at time CLOCK= 0.

Inter-arrival times

15	26	40	6	31	49

Service times

Kate	52	77	84	62
Jane	42	35	43	66

Continue the simulation until five customers have been served. [6]

Question 12

An Air Niugini jet, carrying three hundred passengers, arrives at Port Moresby International Airport, Papua New Guinea. The passengers eventually arrive at passport control with an inter-arrival time given by a normal distribution with mean ten seconds and standard deviation three seconds. They must clear passport control through one of five booths. There are four types of passenger: Papua New Guinea citizens, foreign residents, foreign tourists and foreign business people. (Assume that there are no transit passengers.) Their percentages, the number of passport control booths allocated, and the distributions of the service times (measured in seconds) are given in the following table:

	%	Number of booths	Service times
Citizen	35	2	$E(20)$
Resident	30	1	$\mathcal{N}(30, 10)$
Tourist	15	1	$U(50, 100)$
Business	20	1	$\mathcal{N}(60, 15)$

There is a single queue for the two citizen booths. Each of the other booths has its own queue.

After passing through passport control, all travellers must then pass through customs. Passengers choose either the green line (80%) or the red line (20%). Five per cent of those who choose the green line are selected for a thorough customs search. Service time distributions (measured in seconds) and the number of customs officers are given in the following table:

Number of officers	Type of search	Service times
1	Thorough	$\mathcal{N}(100, 50)$
1	Green	$E(10)$
1	Red	$\mathcal{N}(40, 10)$

After leaving customs, all passengers from the red line, and 20% of those from the green line who were subjected to a thorough search, go to a cashier to pay the customs duty, which takes a time according to a normal distribution with mean 40 seconds and standard deviation 10 seconds.

(i) Construct a queuing diagram of the immigration system. [3]

(ii) Create a SIMIAN model of the immigration system at the Port Moresby International Airport. [9]

(iii) Describe briefly an experiment you would perform to provide the airport authorities with appropriate information on the build-up of queues that they can use to decide on an appropriate allocation of staff to the immigration system for future arrivals of Air Niugini jets. [3]

[END OF QUESTION PAPER]