M371/B

The Open University

Third Level Course Examination 1996
Computational Mathematics

Monday, 14 October, 1996 2.30 pm – 5.30 pm

Time allowed: 3 hours

This paper is divided into **FOUR** parts.

Attempt **SIX** questions of which **NOT MORE THAN TWO** questions should be from any one part of the paper. All questions carry equal weight and are marked out of 15. The final score is out of 90.

Two sheets of graph paper are provided with this question paper for use, if required, in answering questions.

At the end of the examination

Check that you have written your name, personal identifier and examination number on **each** answer book used. **Failure to do so will mean that your work cannot be identified.** Attach all your books together using the paper fastener provided.

Enter the numbers of the questions that you have attempted in the boxes on the front page of the answer book(s).

PART I

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 1

(a) Consider the equation

$$e^x \sin x = 3.$$

(i) Determine the number of positive real roots of this equation in the interval $[0, 2\pi]$. [2]

(ii) Show that the iterative scheme

$$x_{r+1} = g(x_r) = -\ln(\tfrac{1}{3}\sin x_r)$$

satisfies the conditions for a contraction mapping on $[1, 1.5]$. Given $x_0 = 1$ what can you deduce about the sequence of iterates? [6]

(b) Consider the equation

$$f(x, c) = e^{cx} \sin x - 7.46 = 0.$$

(i) For $c = 1$, determine a root of this equation close to $x = 2.3$ correct to three decimal places using the Newton–Raphson method. [3]

(ii) Determine the absolute and relative condition numbers for the problem of finding the root in (b)(i) given that the parameter c is subject to small changes about the value $c = 1$. Hence comment on the conditioning of this problem. [4]

Question 2

Consider the system of linear equations $\mathbf{Ax} = \mathbf{b}$ given by

$$\begin{bmatrix} 1.1 & 2.2 & 3.3 \\ 2.2 & 3.3 & 4.4 \\ 3.3 & 4.4 & 5.51 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 77.7 \\ 88.8 \\ 99.9 \end{bmatrix}.$$

(i) Determine the exact **LU** decomposition of **A**, and hence explain how partial pivoting might help when the Gaussian elimination method, with three-figure arithmetic, is applied to the system of equations. [5]

(ii) The system was solved using the Gaussian elimination method with partial pivoting, together with four-figure arithmetic, to obtain

$$\overline{\mathbf{x}} = [-46.24,\ 52.02,\ 4.286]^T.$$

One iterative refinement was performed to give

$$\mathbf{r} = [0.0238,\ -0.0036,\ 0.01186]^T;$$
$$\delta\mathbf{x} = [3.013,\ -6.093,\ 3.064]^T;$$
$$\mathbf{x}_{\text{new}} = [-49.25,\ 58.11,\ 1.222]^T.$$

Use this information to comment on the conditioning of the problem and on the stability of the method. [7]

(iii) The element $a_{3,3}$ of **A** is changed from 5.51 to 5.5. Explain what goes wrong when the Gaussian elimination method is used to try to solve the new system. [3]

Question 3

(a) Consider the system of linear equations

$$\begin{bmatrix} 2.1 & 8.4 & 3.6 \\ 1.7 & 4.8 & 7.3 \\ 9.2 & 4.0 & 5.1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6.4 \\ 1.1 \\ 3.9 \end{bmatrix}.$$

(i) Rearrange the system in a way such that the iterates are guaranteed to converge when the system is solved using the Gauss–Seidel method. [2]

(ii) Write down the Gauss–Seidel iterative scheme for your rearranged system. [3]

(b) Consider the system of nonlinear equations:

$$f_1(\mathbf{x}) = x_1 - x_2^2 - \sin x_2$$
$$f_2(\mathbf{x}) = x_1^2 \sin x_1 - x_2$$

This system has many roots for $x_1 \geq 0$.

(i) By drawing a sketch or otherwise, show that there are exactly four roots in the region

$$R = \{\mathbf{x} : 0 \leq x_1 \leq 5,\ -5 \leq x_2 \leq 5\}.$$ [3]

(ii) One of the roots in R is near the point $[1,\ 0.5]^T$. Perform one iteration of the Newton–Raphson method, with $\mathbf{x}^{(0)} = [1,\ 0.5]^T$ to determine $\mathbf{x}^{(1)}$. [7]

PART II

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 4

The artisans of a Mexican village produce paintings on earthenware pots, paintings on amate (rough bark paper), and carvings from the thorns of the pochote tree. They transport their produce once a week to the city for sale, but must store it meanwhile in a limited space of 5 cubic metres. They have available between them 1000 man-hours per week. The storage space and labour time required, and the profit, for each painting or carving, are shown in the following table.

	Storage space (m^3)	Labour time (man-hours)	Profit ($)
Painted pot	0.03	3	9
Amate painting	0.002	4	10
Pochote carving	0.005	5.5	16

They wish to maximize their weekly profit.

(i) Formulate their problem as a linear programming model in standard form. [3]

(ii) Write down the dual of the model in (i), and solve it graphically. [6]

(iii) Interpret the dual model and its solution, and deduce the solution to the primal model. [6]

Question 6

Consider the following linear programming model:

$$\begin{aligned}
&\text{minimize } z = x_1 + 3x_2 - x_3\\
&\text{subject to}\\
&3x_1 + x_2 + 4x_3 \geq 10\\
&\quad 4x_2 + x_3 = 6\\
&x_1 + 2x_2 \quad \leq 5\\
&x_1, x_2, x_3 \geq 0
\end{aligned}$$

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(i) Give the augmented form of this model using matrices. [3]

(ii) What is the all-slack solution for this model? Is it feasible? If not, what pseudo-objective and initial pricing vector should be used for the first iteration of the revised simplex method? [4]

(iii) Complete one iteration of the revised simplex method and find a new basic solution. Is this new solution feasible? [5]

(iv) The problem was solved on the computer and the following results were obtained:

Objective function = 5.904 76

Variable	Value	Reduced cost	Constraint	Slack	Shadow price
1	2.238 095	0	1	1	−0.238 095
2	1.380 952	0	2	0	−0.047 619
3	0.476 191	0	3	0	1.714 286

Sensitivity analysis

Right-hand-side values:

Constraint	Shadow price	Lower limit	Given value	Upper limit
1	−0.2381	7.5	10	39
2	−0.0476	4	6	11.88
3	1.714	1.867	5	5.833

Costs:

Variable	Lower limit	Given value	Upper limit
1	0.875	1	3.5
2	−2	3	3.25
3	−1.79E+308	−1	−0.8

What would be the effect on the optimal value of the objective function of each of the following changes:

(a) changing the right-hand side of the second constraint from 6 to 4?

(b) changing the cost coefficient of variable x_2 from 3 to 2? [3]

Question 6

Washing machines are assembled by the Dayline company at its two factories in Northampton and Swindon. It must make weekly deliveries to three retail outlets in Birmingham, Bristol and Cambridge. The following table gives the distribution costs (in pounds per machine) from each factory to each outlet, together with the maximum weekly production at each factory, and the fixed weekly requirement at each outlet.

Table of distribution costs (pounds per machine)

Factory / Outlet	Northampton	Swindon	Fixed requirement (machines per week)
Birmingham	18	12	400
Bristol	20	10	250
Cambridge	15	20	200
Maximum production (machines per week)	640	440	

Let x_{ij} be the number of machines per week from factory i ($i = 1, 2$) to outlet j ($j = 1, 2, 3$).

(i) Write down constraints to ensure that:

(a) maximum production at each factory is not exceeded;

(b) the requirement at each outlet is satisfied exactly.

Complete the formulation of an integer programming model in which the objective is to minimize the distribution costs. [4]

(ii) Dayline has decided to make all deliveries to Cambridge from their Northampton factory. In addition, deliveries to all outlets must be made in full loads of 50 machines. Use this information to revise your model from (i). Your revision should include the elimination of the variables x_{13}, x_{21}, x_{22} and x_{23} from the model. Write down the complete revised model. (It may help to define $x_{11} = 50X_{11}$, etc.) [4]

(iii) Use the branch-and-bound method to solve your revised integer programming problem from (ii) to determine the number of machines to be delivered from each factory to each outlet. Solve each continuous problem with the aid of a graph, and show the details of the solution process in a table of the following kind: [7]

Problem, I_k	Problem solved, C_k	Optimal solution to C_k			Current bound	Problem to pursue	Stored problems
		X_{11}	X_{12}	z			

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 7

Consider the function

$$f(x) = x^2 \cos x.$$

(i) Show that the function has three local minimizers in the interval $[-5, 5]$. Show that one of these is at $x = 0$. How are the other two linked? [5]

(ii) Show that f is unimodal on the interval $[3, 5]$. Starting with this subinterval, perform three iterations of the golden section search method to find a smaller interval containing a local minimizer. Write down this smaller interval. [6]

(iii) How many iterations of the method in (ii) would be required to determine the local minimizer in $[3, 5]$, correct to three decimal places? How many function evaluations would be needed? [3]

(iv) Explain how the golden section search method can be useful when minimizing a function of more than one variable. [1]

Question 8

(i) Complete one iteration of the rank-one method, starting from $\mathbf{x}^{(0)} = [0, 0]^T$ and with $\mathbf{H}^{(0)} = \mathbf{I}$, in order to find an approximation to a local minimizer for the function

$$f(\mathbf{x}) = x_1^4 + 2x_1^2x_2^2 + 2x_2^4 + 2x_1x_2 + x_2.$$

At the end of this iteration, update the approximate inverse Hessian matrix, and hence determine the new search direction. [13]

(ii) Give one disadvantage of the rank-one method applied to general minimization problems for a function of several variables. How does the Davidon–Fletcher–Powell method overcome this disadvantage? [2]

Question 9

Consider the following problem:

$$\text{minimize } f(\mathbf{x}) = \tfrac{3}{2}x_1^2 + 3x_1x_2 + 2x_2^4 - \tfrac{32}{3}x_2^3 - 11x_1 + 4x_2$$
$$\text{subject to } c(\mathbf{x}) = 3x_1 + 2x_2 - 14 = 0$$

(i) Write down the Lagrangian function for this problem. [1]

(ii) Show that $\boldsymbol{\alpha} = [6, -2]^T$ and $\boldsymbol{\alpha} = [4, 1]^T$ are both constrained stationary points and classify them. [12]

(iii) Estimate the new value of f close to $\boldsymbol{\alpha} = [6, -2]^T$ if the constant in the constraint changes from 14 to 13.9. [2]

PART IV

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 10

Consider the definite integral

$$I = \int_0^1 \frac{\cos x}{1+x}\,dx$$

which is to be estimated using the six normalized pseudo-random numbers

0.797	0.360	0.007	0.881	0.092	0.280

(i) Use the crude Monte Carlo method to estimate the integral, I, and determine a 95% confidence interval for your estimate. [5]

(ii) Use antithetic variables and the crude Monte Carlo method to obtain a second estimate for I. [3]

(iii) Using antithetic variables and the crude Monte Carlo method, with a sample of size 50, a computer gave the following results, where $f(x) = \cos x/(1+x)$:

$$\sum_{i=1}^{50} f(x_i) = 30.304\,13 \qquad \sum_{i=1}^{50} f(1-x_i) = 30.013\,70$$

$$\sum_{i=1}^{50} (f(x_i) + f(1-x_i))^2 = 72.814\,358$$

Use these results to determine an estimate for the integral and a 95% confidence interval for this estimate. Estimate the sample size required to obtain the value of I correct to three decimal places. [7]

Question 11

This question concerns the operation of the Tampico Sea-Food Restaurant, which has 16 tables available, where each table is large enough to cater for any group of customers.

(i) Groups of customers arrive at the restaurant with inter-arrival times distributed according to a negative exponential distribution with mean 3 minutes. Sample from this distribution, using the following four normalized pseudo-random numbers, to obtain four inter-arrival times, and the corresponding arrival times, all rounded to the nearest second.

0.316	0.562	0.923	0.120

[3]

(ii) The time taken for a group to eat its meal is normally distributed with mean 40 minutes and standard deviation 10 minutes. Use the following four normalized pseudo-random numbers to sample from this normal distribution, rounding your answers to the nearest minute.

0.537	0.980	0.197	0.158

[3]

(iii) In addition, each group takes five minutes to order its meal and be served, and another two minutes to obtain its bill and pay it (after eating its meal and before freeing its table). Will there be congestion to obtain tables? Would it be possible to reduce the number of tables, or would you recommend making space for more tables? [3]

(iv) After a group of customers has found a table, it must obtain the attention of one of three waitresses, who will devote herself to that group to take its order and serve it. After completing its meal, the group must again obtain the attention of one of the waitresses to obtain its bill and pay it. Write a SIMIAN model which simulates the operation of the restaurant, second by second. [6]

~~Question 12~~

A Gallic Airways aeroplane lands at Prestheath Airport after a flight from Mauritius. Once the plane has safely come to a halt, the 200 passengers disembark, before going through Passport Control, Baggage Reclaim and Customs.

At Passport Control there are two desks for EU nationals, at which a single queue forms, and one desk for non-EU passport holders, which has its own queue. 80% of passengers hold EU passports. Passengers reclaim their luggage from a single carousel.

90% of passengers go through the green channel at Customs; the remaining 10% have something to declare and use the red channel. 5% of the passengers using the green channel are stopped by customs officials to have their baggage inspected.

The 20 cabin crew disembark together once the last passenger has done so, and arrive as a group at Passport Control about 15 minutes after the first passenger arrives there. They are all EU passport holders and are allowed to go to the front of the queue for the EU passport desk. They do not have any baggage to reclaim. Their pattern of passage through Customs is the same as for ordinary passengers.

By the time they reach Passport Control, the passengers arrival rate is random, with a mean of about 5 seconds. The service time for EU nationals at Passport Control is normally distributed with a mean of 20 seconds and a standard deviation of 5 seconds; that for non-EU nationals is normally distributed with mean 40 seconds and standard deviation 20 seconds. The time for reclaiming baggage is uniformly distributed between 5 and 15 minutes. It takes passengers 1 minute to go through the green channel, unless they are stopped, in which case they take a time that follows a negative exponential distribution with mean 10 minutes. Passengers going through the red channel take a time that follows a normal distribution with mean 5 minutes and standard deviation $2\frac{1}{2}$ minutes.

(i) Construct a SIMIAN model of the arrival of the passengers and crew at Prestheath (using a suitable time unit). [*13*]

(ii) Explain briefly how you might extend your model to cover the arrival of a number of aeroplanes of different sizes at Prestheath over a 24-hour period. [*2*]

[END OF QUESTION PAPER]