M371/J

The Open University

Third Level Course Examination 1995

Computational Mathematics

Friday, 20 October, 1995 10.00 am – 1.00 pm

Time allowed: 3 hours

This paper is divided into **FOUR** parts.

Attempt **SIX** questions of which **NOT MORE THAN TWO** questions should be from any one part of the paper. All questions carry equal weight and are marked out of 15. The final score is out of 90.

Two sheets of graph paper are provided with this question paper for use, if required, in answering questions.

At the end of the examination

Check that you have written your name, personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach all your books together using the paper fastener provided.

Enter the numbers of the questions that you have attempted in the boxes on the front page of the answer book(s).

PART I

You should attempt to answer **not more than TWO** *questions from this part and not more than six questions overall.*

Question 1

Consider the equation

$$x^3 = \exp(1/x).$$

(i) Determine the number of positive real roots of this equation. [3]

(ii) Show that the iterative scheme

$$x_{r+1} = g(x_r) = \exp(1/(3x_r))$$

satisfies the conditions for a contraction mapping on [1, 2] and deduce that if x_0 is in this interval then the iterates will converge to a root of the above equation. Starting with $x_0 = 1.75$, use the iterative scheme to determine a root α of this equation correct to two decimal places. [8]

(iii) Let

$$f(x, c) = cx^3 - \exp(1/x),$$

where the parameter c is subject to small changes about the value $c = 1$. Determine the absolute and relative condition numbers for the problem of finding the root α of the equation $f(x, c) = 0$ and comment on the conditioning of this problem. [4]

Question 2

Consider the system of linear equations $\mathbf{Ax} = \mathbf{b}$ given by

$$\begin{bmatrix} -8 & 6 & 3 \\ -3 & 2 & 1.123 \\ -8.5 & 6 & 3.202 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 6 \end{bmatrix}.$$

Using four-figure arithmetic and the Gaussian elimination method, the approximate **LU** decomposition of **A** is given by

$$\overline{\mathbf{L}} = \begin{bmatrix} 1 & 0 & 0 \\ 0.375 & 1 & 0 \\ 1.062 & 1.488 & 1 \end{bmatrix} \qquad \overline{\mathbf{U}} = \begin{bmatrix} -8 & 6 & 3 \\ 0 & -0.25 & -0.002 \\ 0 & 0 & 0.01898 \end{bmatrix}$$

and the approximate solution by

$$\overline{\mathbf{x}} = [123.3, \ -22.98, \ 372.0]^T.$$

(i) Use the full precision of your calculator to compute the residual vector $\mathbf{r}$, and to carry out one iterative refinement to obtain a revised solution. Round your solution to four decimal places. [6]

(ii) Use your results from (i) to comment on the conditioning of the problem and on the stability of the method. [4]

(iii) Determine the absolute and relative condition numbers, k_a and k_r, for the problem, using (for this part only) the information that

$$\mathbf{A}^{-1} = \begin{bmatrix} -9.5429 & -34.6285 & 21.0857 \\ 1.7286 & -3.3143 & -0.4571 \\ -28.5714 & -85.7142 & 57.1428 \end{bmatrix}$$

correct to four decimal places. Comment on your results. [5]

Question 3

This question concerns the system of equations

$$f_1(x_1, x_2) = x_1^2 - 5x_2 + 7x_1 = 0$$
$$f_2(x_1, x_2) = x_1 + 2x_2^2 - 1 = 0$$

which is to be solved in the region

$$R = \{\mathbf{x} : 0 \le x_1 \le 0.8, 0 \le x_2 \le 1\}.$$

(i) Consider the following simple iteration scheme for the system:

$$x_1^{(r+1)} = g_1(\mathbf{x}^{(r)}) = \frac{5x_2^{(r)}}{(x_1^{(r)} + 7)}$$
$$x_2^{(r+1)} = g_2(\mathbf{x}^{(r)}) = \sqrt{\tfrac{1}{2}(1 - x_1^{(r)})}$$

Use the Contraction Mapping Theorem, with this scheme, to show that the system has a unique solution in R. [5]

(ii) Perform **two** iterations of the Newton–Raphson method, starting from $\mathbf{x}^{(0)} = \mathbf{0}$, to find an approximation to the solution of the system $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. [6]

(iii) State two difficulties with the Newton–Raphson method for large systems of equations. Describe briefly how the add-**Nx** method overcomes these difficulties, stating briefly how it overcomes them. What drawbacks does this method have? [4]

PART II

You should attempt to answer **not more than TWO** *questions from this part and not more than six questions overall.*

Question 4

A computer chip manufacturer produces three types of chip: Pentia, Powerplus and Ultima. After basic production, there are two assembly processes. Only the Ultima chip needs both of these. The processes required, the selling price, and the maximum weekly production capacities are shown in the following table.

Process	Basic production	Assemble I	Assemble II	Selling price (£ per chip)
Pentia	✓	×	✓	7.50
Powerplus	✓	✓	×	10.00
Ultima	✓	✓	✓	12.50
Maximum capacity (per week)	12 000	8000	6000	

The demands for Powerplus and Ultima chips are unlimited, but no more than 900 Pentia chips can be sold per week. The manufacturer wishes to maximize her weekly income.

(i) Formulate this problem as a linear programming model in standard form. [4]

(ii) Write down the dual of the model in (i). [2]

(iii) *Without computing its solution*, show that the primal problem (in (i)) has an optimal solution. [3]

(iv) The following optimal solution to the problem was found using the simplex method on a computer:

Objective function: 99 500

Variable	Name	Value	Reduced cost
1	Pentia	900	0
2	Powerplus	2900	0
3	Ultima	5100	0

Constraint	Name	Slack	Shadow price
1	Production	3100	0
2	Assemble I	0	10
3	Assemble II	0	2.5
4	Pentia sales	0	5

(a) Use the solution to determine the effect on the optimal value of the objective function of increasing the maximum weekly capacity of the basic production process from 12 000 to 12 500.

(b) Use this solution to determine the effect on the optimal value of the objective function of increasing the maximum weekly production capacity of the Assemble I process from 8000 to 8500, stating any assumption you make.

(c) Give an economic interpretation of the zero reduced cost values. [6]

Question 5

Consider the following linear programming model:

$$\begin{aligned} &\text{minimize } z = x_1 - 3x_2 + 3x_3 + 5x_4 \\ &\text{subject to} \\ &4x_1 - x_2 + 2x_3 + 2x_4 \le 9 \\ &-3x_1 + 3x_2 + 2x_3 = 8 \\ &2x_1 - 2x_2 + x_3 + 2x_4 \ge 6 \\ &x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

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(i) Give the augmented form of this model using matrices. [3]

(ii) What is the all-slack solution for this model? Is it feasible? If not, what pseudo-objective and initial pricing vector should be used for the first iteration of the revised simplex method? [4]

(iii) Complete **one** iteration of the revised simplex method and find a new basic solution. Is this new solution feasible? [5]

(iv) The problem was solved on the computer and the following results were obtained:

Objective function = 19.5

Variable	Value	Reduced cost	Constraint	Slack	Shadow price
1	0	5.5	1	1	2.5
2	2	0	2	0	−1.5
3	1	0	3	0	−5
4	4.5	0			

Sensitivity analysis

Right-hand-side values:

Constraint	Shadow price	Lower limit	Given value	Upper limit
1	2.5	8.667	9	10
2	−1.5	6	8	9
3	−5	5	6	6.333

Costs:

Variable	Lower limit	Given value	Upper limit
1	−4.5	1	1.79E+308
2	−3.786	−3	1.79E+308
3	−1.79E+308	3	3.611
4	4.476	5	1.79E+308

What would be the effect on the optimal value of the objective function of each of the following changes:

(a) changing the right-hand side of the second constraint from 8 to 9?

(b) changing the cost coefficient of variable x_1 from 1 to 2? [3]

Question 6

The Skyways television assembly factory is planning a short production run of its Skycam model, which is produced in two versions: Standard and Deluxe. Production involves three processes: assembly, testing and delivery. The following table gives the time in person-hours for each process and for each version, the total number of person-hours available for each process, and the sale price for each version (in thousands of ECUs).

Process	Standard	Deluxe	Time available
Assembly (person-hours)	8	5	38
Testing (person-hours)	3	6	24
Delivery (person-hours)	6	7	32
Sale price (1000s ECUs)	19	26	–

(i) Formulate the problem as an integer programming model, given that the factory wishes to maximize its sales' receipts for this production run. [4]

(ii) Solve the problem using the branch and bound method, stating clearly the branching strategy used, and hence determine the quantity of each version the factory should produce. Solve each continuous subproblem graphically and show the details of the solution process in a table of the the following kind:

Problem, I_k	Problem solved, C_k	Optimal solution to C_k			Current bound	Problem to pursue	Stored problems
		x_1	x_2	z			

[8]

(iii) Extend your model in (i), using a 0–1 variable, in order to be able to use it to determine whether it would be worth replacing the Standard version by a new Hitech version, for which the assembly time is 5 person-hours, the testing time is 7 person-hours, the delivery time is 9 person-hours and the sale price is 32 000 ECUs. [3]

PART III

You should attempt to answer **not more than TWO** *questions from this part and not more than six questions overall.*

Question 7

(i) Describe briefly the search strategies of the alternating variables and steepest descent methods for finding a local minimizer for a function of two variables. [2]

(ii) Perform **one** iteration of **each** method in (i), starting from $\mathbf{x}^{(0)} = [0, 0]^T$, in order to find an approximate local minimizer for the function

$$f(\mathbf{x}) = 2x_1^4 + 2x_1x_2 + 2x_2^2 - x_1 - 3x_2.$$

The line searches should be performed analytically. In each case state the search direction that would be used for the second iteration. [9]

(iii) Explain briefly how the golden section search and quadratic search methods are used to find a local minimizer for a function f which is unimodal on a starting interval $[a, b]$ in which a local minimizer is known to lie. [4]

Question 8

(i) Carry out **one** iteration of the Newton–Raphson method, starting from $\mathbf{x}^{(0)} = [\frac{1}{2}, \frac{1}{2}]^T$, in order to find an approximate local minimizer for the function

$$f(\mathbf{x}) = 5x_1^4 - 2x_1x_2 + 7x_2^2 - 3x_1 - 5x_2.$$ [7]

(ii) Mention three disadvantages of the Newton Raphson method for finding a local minimizer and explain briefly how these disadvantages can be overcome. [6]

(iii) Why might you use inexact line searches with an optimization method? [2]

Question 9

Consider the following problem:

$$\text{minimize } f(\mathbf{x}) = 2x_1^2 - 3x_1x_2 + x_2^2 + 13x_1 - x_2$$
$$\text{subject to } c(\mathbf{x}) = 4x_1 + x_2 = 7$$

(i) Write down the Lagrangian function for this problem. [2]

(ii) Show that there is a constrained stationary point at $\boldsymbol{\alpha} = [1, 3]^T$ and classify this stationary point. [10]

(iii) Estimate the new minimum value of f if the right-hand side of the constraint changes from 7 to 7.1. [3]

PART IV

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

~~Question 1~~0

Consider the following definite integral:

$$\int_0^2 x^2 \exp(-x)dx.$$

(i) (a) Use the formula

$$x_i = ax_{i-1} + b \pmod{m}$$

to produce five pseudo-random numbers, $r_0, r_1, \ldots, r_4$, on $[0, 2]$ using

$$x_0 = 5, a = 11, b = 3, m = 17.$$ [3]

(b) Use these numbers to estimate the integral by the crude Monte Carlo method. [3]

(c) Find a 95% confidence interval for your estimate. [2]

(ii) (a) Given that

$$g(x) = \tfrac{1}{3}x$$

is a suitable regular part for the integral, use the following pseudo-random numbers on [0, 2] to estimate the integral by the method of extraction of the regular part:

0.6588	0.5496	1.8458	1.5090	1.0178

[5]

(b) Find a 95% confidence interval for your estimate. [2]

Question 11

This question concerns the operation of the Nisaru Motor Spares supply depot to which motorists come to purchase car spares and oil.

(i) Motorists arrive at the depot with inter-arrival times distributed according to a negative exponential distribution with mean 200 seconds. Sample from this distribution, using the following five normalized pseudo-random numbers, to obtain five inter-arrival times and hence five arrival times, all rounded to the nearest second.

0.764	0.270	0.727	0.120	0.533

[3]

(ii) Motorists first queue for service with the car-spares server, whose service time is normally distributed with mean 150 seconds and standard deviation 50 seconds. Use the following four normalized pseudo-random numbers to sample from this distribution.

0.163	0.268	0.112	0.537

[3]

(iii) What is the expected queue length (including the motorist being served) and the expected waiting time (including the service time) for the car-spares server? [3]

(iv) After collecting their car spares, some of the motorists proceed to an oils distributor (where a zero oil-distributor service time indicates that that motorist does not require oil, and so does not join the queue for the oil distributor). Use the following data to perform a hand simulation of the operation of Nisaru Motor Spares supply depot, up to and including the arrival of the sixth motorist.

Inter arrival times (secs)	250	56	243	41	209	165
Car-spares server service times (secs)	87	121	75	162	249	133
Oil distributor service times (secs)	305	0	281	326	0	0

[6]

Question 12

Tickets for a test match series are issued by the Ashtour Ticket Agency, which employs five people: a manager, two ticket clerks, a receptionist and a cashier. The Agency has three types of customer: personal callers, telephone callers and letter writers. The Agency is open for nine hours daily, from 8 am until 5 pm.

Personal callers are given priority over telephone callers who are given priority over letter writers. Personal callers arrive with inter-arrival times given by a negative exponential distribution with mean 8 minutes. Telephone calls arrive with inter-arrival times given by a negative exponential distribution with mean 9 minutes. The delivery of letters, totalling 100 a day, arrives before the Agency opens for business.

All letters are dealt with in the first instance by the two clerks; 20% of these letters are then handed on to the manager and 70% to the cashier; the remaining 10% require no further attention from any of the Agency's staff.

All telephone callers are dealt with firstly by the receptionist, taking 2 minutes for each call, and she completes the service for 15% of such customers: the manager then deals with 10% and the clerks with the rest (75%). Personal callers are first dealt with by the receptionist, taking 2 minutes, and she completes the service for 10% of such customers; the manager then deals with 20% and the clerks with the rest (70%).

If the manager is busy when required by a customer, the receptionist asks personal callers to wait and telephone callers to ring back; all customers to be dealt with by a ticket clerk are asked to wait if no clerk is free. The manager takes a time distributed according to a normal distribution with mean 10 minutes and standard deviation 8 minutes to deal with each customer. 40% of customers dealt with by the manager are referred by her to the cashier so that they can make a payment; the remaining 60% require no further attention from any of the Agency's staff.

The time taken by the ticket clerks to deal with a customer is distributed according to a normal distribution with mean 5 minutes and standard deviation 3 minutes. All personal and telephone callers dealt with by the clerks are passed on to the cashier, as are 80% of letters (see above).

All telephone callers pay the cashier by credit card. Half of personal callers pay the cashier by credit card, the other half by cash or cheque. All letter writers pay by cheque. Cash and cheque payments both take 1 minute per customer. Credit card payments take a time that varies according to a normal distribution with mean 3 minutes and standard deviation 2 minutes.

Personal callers form four separate single queues for the receptionist, the ticket clerks, the manager and the cashier. Telephone callers form a single telephonic queue for the receptionist, and are queued telephonically by the receptionist into a single queue for the clerks and a single queue for the cashier; telephone callers do not form a queue for the manager as they are turned away if she is busy (see above). Letters are dealt with on a first-in first-out basis by all staff of the Agency.

(i) Draw a queuing diagram for the flow of *letters* through the Agency. [3]

(ii) Write a SIMIAN model of the daily routine of the Ashtour Ticket Agency, from the time the Agency opens until the time that the last customer of the day has been served. You may assume that customers (both personal and telephone callers) who call outside the Agency's opening hours are turned away and that none of the Agency's staff takes lunch or other breaks. [12]

[END OF QUESTION PAPER]