M371/S

The Open University

Third Level Course Examination 2000

Computational Mathematics

Wednesday, 18 October, 2000 2.30 pm – 5.30 pm

Time allowed: 3 hours

This paper is divided into **FOUR** parts.

Attempt **SIX** questions of which **NOT MORE THAN TWO** questions should be from any one part of the paper. All questions carry equal weight and are marked out of 15. The final score is out of 90.

Two sheets of graph paper are provided with this question paper for use, if required, in answering questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach all your books together using the paper fastener provided.

Enter the numbers of the questions that you have attempted in the boxes on the front page of the answer book(s).

PART I

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 1

Consider the equation

$$e^x = 5\cos x$$

on the interval $[0, 1.5]$.

(i) Show that there is one root of the equation in this interval. [2]

(ii) Use the Newton–Raphson method to find this root correct to five decimal places, starting from $x_0 = 1$. [3]

(iii) Show that the iterative scheme

$$x_{r+1} = \ln(5\cos x_r)$$

does not satisfy either of the conditions of the Contraction Mapping Theorem on the interval $[0, 1]$. [5]

N/A (iv) Consider the iterates generated by the iterative scheme $x_{r+1} = g(x_r)$. Given that the displacement ratio $\lambda \simeq \Delta_r/\Delta_{r-1}$, where $\Delta_r = x_{r+1} - x_r$, is constant for sufficiently large r, derive the Aitken acceleration formula

$$x_{\text{new}} = x_r + \tfrac{\lambda}{1-\lambda}\Delta_{r-1}.$$

What restriction, if any, is there on the magnitude of λ? Perform three iterations using the iterative scheme $x_{r+1} = \ln(5\cos x_r)$, starting from $x_0 = 1$, followed by one application of the Aitken acceleration formula (ignoring any restriction on the magnitude of λ). Comment on your results. [5]

Question 2

Consider the system of linear equations $\mathbf{Ax} = \mathbf{b}$ given by

$$\begin{bmatrix} 8.9 & 0.98 & 0.78 \\ 3.2 & 2.5 & 0.703 \\ 4.77 & 5.0 & 1.3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0.075 \\ 0.5 \\ 0.13 \end{bmatrix}$$

(i) Solve the problem using the Gaussian elimination method with partial pivoting and the full accuracy of your calculator. [7]

(ii) Using four-figure arithmetic, we obtain a solution $\hat{\mathbf{x}} = [35.45,\ 105.9,\ -537.4]^T$. One iterative refinement gives residual vector $\mathbf{r} = [0.04,\ -0.1022,\ -0.1535]^T$ and $\delta\mathbf{x} = [-2.682,\ -8.076,\ 40.79]^T$, so that the improved solution, correct to four decimal places, is

$$\mathbf{x}_{\text{new}} = [38.13,\ 114.0,\ -578.2]^T.$$

(a) Determine the new residual vector for this improved solution using the full accuracy of your calculator. [3]

(b) Use the given information to comment on the absolute and relative conditioning of the problem and on the stability of the method. [5]

Question 3

Consider the following system of non-linear equations:

$$f_1(x_1, x_2) = x_1^3 + \tfrac{1}{3}x_2^3 - 1 = 0$$
$$f_2(x_1, x_2) = 5x_1x_2 - 3 = 0$$

(i) Determine, graphically or otherwise, the number of roots of the system such that $x_1 \geq 0$ and $x_2 \geq 0$, and their approximate locations. [3]

(ii) Write down the Gauss–Jacobi iterative scheme for this system of equations in the order given, and show that it is a contraction mapping on the region

$$R = \{(x_1, x_2) : 0.8 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}.$$ [9]

(iii) Carry out **three** iterations of the Gauss–Jacobi method to find an approximate root of the system, starting from $\mathbf{x}_0 = [1, 1]^T$. [3]

PART II

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 4

A wholesale IT company, Netco.com, wants to buy the entire stock of a small rival company Itco.com. Netco.com wishes to determine how much it needs to pay Itco.com for each of the items Itco.com has in stock, i.e. mobile phones, printers, standard computers and premium computers, in order to minimize its total outlay. However, Itco.com refuses to sell unless the offer from Netco.com at least matches its current selling prices to retail shops. Itco.com sells to its retail shops in made-up lots X and Y as shown below.

	Retail price (£000)	Mobile phones	Printers	Standard computers	Premium computers
Lot X	16	2	6	6	2
Lot Y	24	3	5	11	5
Total stock		225	540	660	380

The prices paid by Netco.com for each of the four products must ensure that Itco.com could not have done better by selling directly to retail shops as Lots X and Y.

(i) Formulate Netco.com's problem as a linear programming model in general form, ignoring any integer constraints. [4]

(ii) State the dual of the model in (i) in standard form. Give meanings to the dual variables and to the objective function. [5]

(iii) Solve the dual problem graphically and interpret the optimal value of the objective function of the dual in terms of the primal model in (i). [6]

Question 5

Consider the linear programming model:

$$\begin{aligned} &\text{maximize } z = x_1 - 3x_2 + 4x_3 \\ &\text{subject to} \\ &3x_1 \qquad\quad + 4x_3 \le 6 \\ &x_1 + 3x_2 + \ x_3 = 5 \\ &3x_1 + 2x_2 - \ x_3 \ge 8 \\ &x_1, x_2, x_3 \ge 0 \end{aligned}$$

(i) Give the augmented form of this model using matrices. [2]

(ii) What is the all-slack solution? Is it feasible? If not, what pseudo-objective and initial pricing vector should be used for the first iteration of Phase I of the revised simplex method, starting from the all-slack solution? [3]

(iii) Complete one iteration of the revised simplex method and find a new basic solution. Is this new solution feasible? Write down the objective or pseudo-objective function for the second iteration of the revised simplex method. [8]

(iv) The problem was solved on the computer and the following optimal solution and sensitivity analysis results were obtained:

Objective function = −1

Variable	Name	Value	Reduced cost
1	VAR1	2	0
2	VAR2	1	0
3	VAR3	0	0

Constraint	Name	Slack	Shadow price
1	CON1	0	1.046 512
2	CON2	0	−0.674 4186
3	CON3	0	−0.488 3721

Sensitivity Analysis

Right-hand-side values:

Constraint number	Constraint name	Shadow price	Lower limit	Given value	Upper limit
1	CON1	1.047	6	6	16.75
2	CON2	−0.6744	5	5	15.75
3	CON3	−0.4884	0.8333	8	8

Costs:

Variable number	Variable name	Lower limit	Given value	Upper limit
1	VAR1	−8	1	2.75
2	VAR2	−24	−3	8.25
3	VAR3	1.667	4	1.79e+308

(a) What would be the effect on the optimal value of the objective function if the right-hand side of the first constraint changes from 6 to 7?

(b) What would be the effect on the optimal solution if the cost of x_1 changes from 1 to 2? [2]

Question 6

The Wantok workshop assembles televisions and mobile phones from kits. The kits are obtained from a wholesaler each morning and, at the end of the day, the assembled appliances are moved to a showroom for sale the next day, leaving the workshop empty overnight. Once started, an assembly must be completed the same day. The total floor space available in the workshop, for assembling the appliances, is 40 square metres. The owner has £300 to buy kits each day and the employees work a total of 22 hours per day. The following table shows the workshop floor space required for each appliance, the purchase cost of each kit, the number of person-hours required for its assembly and the profit obtained from the sale of the appliance.

	Floor space (m^2)	Purchase price (£)	Time (person-hours)	Profit (£)
Televisions	5	70	4.5	45
Mobile phones	8	45	4	30
Total available	40	300	22	

(i) Formulate this problem as an integer programming model with the objective of maximizing daily profit, given that the workshop can sell all that it assembles. [3]

(ii) Solve the problem using the branch-and-bound method to determine how many kits of each type should be assembled each day, using a suitable branching strategy that you should specify. Solve each continuous subproblem graphically. Construct a table of your results along the following lines. [9]

Problem, I_k	Problem solved, C_k	Optimal solution to C_k			Current bound	Problem to pursue	Stored problems
		x_1	x_2	z			

(iii) Extend your model in (i) using a 0–1 variable so that it can be used to determine whether or not to abandon the assembly of mobile phones in favour of assembling vacuum cleaners. The assembly of vacuum cleaners requires a floor space of $6\,m^2$, each kit costs £50, requires 2 person-hours for assembly and attracts a profit of £90. **You are not required to solve this extended model.** [3]

PART III

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 7

Consider the function

$$f(x) = 5x^2 - 1.25e^x \sin x.$$

(i) Find an interval $[n, n+1]$ that contains a local minimizer, for some integer n. [3]

(ii) Perform **one** iteration of the quadratic search method to determine a smaller interval containing this local minimizer. [6]

(iii) How many function evaluations would be needed by the golden section search method to reduce the interval length from 1 to 0.005? [3]

(iv) What advantage does the golden section search method have over the quadratic search method? Why do we still use the quadratic search method? [3]

Question 8

(i) Carry out **one** iteration of the Newton–Raphson method (without line searches), starting from $\mathbf{x}^{(0)} = [1, 1]^T$, to find an approximation to a local minimizer of the function

$$f(x_1, x_2) = 10x_1^4 + 5x_1x_2^2 - 6x_1^2 - 5x_2^2 + 4x_2^4.$$ [10]

(ii) Describe **four** drawbacks of the Newton–Raphson method (without line searches) as a method of finding local minimizers. In each case explain how the drawback can (at least partially) be overcome either by modifying the Newton–Raphson method or by using a different method. [5]

Question 9

Consider the following constrained minimization model:

$$\text{minimize } f(\mathbf{x}) = 2x_1^4 + 3x_1^2 + x_2^2 - 4x_2x_3 - 3x_1x_3 + 44x_1 + 17x_2 + 38x_3$$

subject to

$$c_1(\mathbf{x}) = 2x_1 + x_2^2 + x_3 - 4 = 0$$

$$c_2(\mathbf{x}) = x_1x_2 + x_3^2 - 2 = 0$$

(i) Write down the Lagrangian function for this model, using the Lagrange multipliers μ_1 and μ_2. [2]

(ii) Show that $\alpha = [-1, 2, 2]^T$ is a constrained stationary point and determine the values of the corresponding Lagrange multipliers. Show further that α is a constrained local minimizer. [13]

PART IV

You should attempt to answer **not more than TWO** *questions from this part and not more than* **six** *questions overall.*

Question 10

Consider the definite integral

$$I = \int_0^2 \frac{1}{1 + e^{\sin x}}\, dx$$

which is to be estimated using the following five normalized pseudo-random numbers on the interval $[0, 2]$:

0.6588	0.5496	1.8458	1.5090	1.0178

(i) Use the crude Monte Carlo method to estimate the integral I, and determine a 95% confidence interval for your estimate. *[6]*

(ii) Show that the function

$$g(x) = 0.1x^2 - 0.3x + 0.5$$

is a suitable regular part for the integral I. Use the method of extraction of the regular part and the crude Monte Carlo method with the same five pseudo-random numbers to obtain another estimate for I together with a 95% confidence interval. *[8]*

(iii) Comment on the results that you obtained in (i) and (ii). *[1]*

Question 11

This question concerns the operation of a small dental clinic, staffed by two dentists, Amanda and Beth, and a receptionist, Colin, from the time that the clinic opens.

(i) Patients arrive with inter-arrival times distributed according to a negative exponential distribution with mean 10 minutes. Sample from this distribution using the following five normalized pseudo-random numbers, to obtain five inter-arrival times and the corresponding arrival times, rounded to the nearest second.

0.123	0.512	0.475	0.087	0.916

[3]

(ii) Patients first queue at the reception desk, where Colin takes their details and checks which dentist they should see. This process takes a time normally distributed with mean 200 seconds and standard deviation 60 seconds. Use the following four normalized pseudo-random numbers to sample from this distribution, rounding your answers to the nearest second.

0.828	0.197	0.522	0.206

[3]

(iii) What is the expected queue length at the reception desk (including the patient being attended to by Colin) and the expected waiting time (including the time being attended to by Colin) at the desk? *[3]*

(iv) Once the patients have seen Colin, they divide into two queues, one for each dentist. Assuming that there are no patients queuing when the clinic opens, use the following data to perform a hand simulation of the operation of the clinic up to and including the arrival of the fifth patient.

Patient number	1	2	3	4	5	6
Inter-arrival times (seconds)	275	70	448	217	2386	1151
Service time by Colin (seconds)	261	223	267	114	422	220
Time with Amanda (seconds)	792	—	1092	—	1260	1296
Time with Beth (seconds)	—	876	—	1296	—	—

[6]

Question 12

A service station has five petrol pumps, each with its own separate queue, and two car washes, for which a single queue forms. It also has a tyre-repair service and a wheel-balancing service, each staffed by one employee and each having its own queue. There are two cashiers with a single queue of customers, who pay for all of their services at one of the cashiers.

65% of all customers come to buy petrol, 20% come to have their cars washed, 10% come for tyre repairs and 5% come for wheel balancing. After having their cars washed, 50% of car-wash customers go on to buy petrol, 20% have a tyre repaired and the other 30% go straight to the cashiers. Of the customers who have a tyre repaired, 60% go on to have their wheels balanced. 40% of customers who have a tyre repaired (including those who have their wheels balanced) go on to buy petrol, as do 40% of those who just have their wheels balanced. Priority for wheel balancing is given to customers who have had a tyre repaired. After buying petrol, customers always go straight to the cashiers.

Customers for petrol always join the shortest queue. Car-wash customers are unwilling to wait for the car wash if the queue (including the cars being washed) is four or more long, and these leave the garage straight away to look for another car wash.

Customers arrive at the garage according to a negative exponential distribution with mean 1 minute. Service times at each petrol pump are normally distributed with mean 3 minutes and standard deviation 1 minute. The service times at the cashiers is normally distributed with mean 2 minutes and standard deviation 1 minute. Car washes take a uniform 10 minutes. The tyre repair time is uniformly distributed between 8 and 12 minutes and the wheel balancing time is uniformly distributed between 6 and 8 minutes.

(i) Create a SIMIAN model of the operation of the petrol station using a time unit of 1 minute. *[12]*

(ii) The manager of the petrol station wants to see whether he can cut back on staff by employing the same person to do the tyre repairs and the wheel balancing. Describe briefly a SIMIAN experiment that would provide the manager with appropriate information, based on a modified version of your model in (i). Your description should include a discussion of suitable performance measures. *(You are not expected to give details of the necessary modifications to the model in (i).)* [3]

[END OF QUESTION PAPER]