Question 5

(a) Find the residues of the function
$$f(z) = \frac{1}{z^3 - 1}$$
at each of the poles of f. [4]

(b) Hence evaluate the real improper integral
$$\int_{-\infty}^{\infty} \frac{1}{t^3 - 1}\, dt.$$
[4]

Question 6

Let $D_0 = \{z : |z| < 2\}$ and $D_1 = \{z : |z| > 2\}$. Show that the analytic functions
$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \qquad (z \in D_0)$$
and
$$g(z) = -\sum_{n=1}^{\infty} \left(\frac{2}{z}\right)^n \qquad (z \in D_1)$$
are indirect analytic continuations of each other. [8]

Question 7

Let $q(z) = 1/\bar{z}^2$ be a velocity function.

(a) Explain why q represents a model fluid flow on $\mathbb{C} - \{0\}$. [1]

(b) Determine a stream function for this flow. Hence find the equation of the streamline through the point i, and sketch this, indicating the direction of flow. [5]

(c) Find the flux of q across the unit circle. [2]

Question 8

(a) Find the fixed points of the function $f(z) = 2z(1 - z)$ and classify them as (super-)attracting, repelling or indifferent. [3]

(b) Which of the following points c lie in the Mandelbrot set:

(i) $c = -1 + i$;

(ii) $c = -1 - \frac{1}{8}i$?

Justify your answer in each case. [5]