## Question 11

(a) (i) Use Rouché's Theorem to show that the equation

$$z^3 + 3z^2 + 1 = 0$$

has two solutions in the disc $\{z : |z| < 1\}$ and one in the annulus $\{z : 1 < |z| < 4\}$.

(ii) Explain why the solution in the annulus is in fact real and negative. *[8]*

(b) (i) Show that

$$\max\{|e^{z^3}| : |z| \leq 3\} = e^{27}$$

and find the point, or points, at which this maximum is attained.

(ii) Hence find an upper estimate for

$$\left| \int_C \frac{\overline{z} + 1}{\overline{z} - 1} e^{z^3} \, dz \right|$$

where $C$ is the circle $\{z : |z| = 3\}$. *[10]*

## Question 12

(a) Determine the Möbius transformation which maps the points $1, 2i, \infty$ to the points $i, \infty, -1$, respectively. *[5]*

(b) (i) Sketch the region

$$\mathcal{R} = \{z : |z| < 1, \operatorname{Im} z > 1 - \operatorname{Re} z\}.$$

(ii) Determine and sketch the image of $\mathcal{R}$ under the Möbius transformation

$$f_1(z) = \frac{z - i}{z - 1}.$$

(iii) Hence determine a one-one conformal mapping which maps $\mathcal{R}$ onto the open upper half-plane. *[13]*

[END OF QUESTION PAPER]