(b) (i) The function g is analytic on $\mathbb{C}-\{0\}$, but not at 0, and so 0 is the only singularity of g. 1M

(ii) We have

$$\begin{aligned} g(z) &= z\cos(1/z^2) \\ &= z\left(1-\frac{1}{2!}\left(\frac{1}{z^2}\right)^2+\frac{1}{4!}\left(\frac{1}{z^2}\right)^4-\cdots\right) && \text{1M} \\ &= z-\frac{1}{2!}\cdot\frac{1}{z^3}+\frac{1}{4!}\cdot\frac{1}{z^7}-\cdots, \qquad \text{for } z\in\mathbb{C}-\{0\}. && \text{1A} \end{aligned}$$

The general term of this series is of the form

$$z\frac{(-1)^n}{(2n)!}\left(\frac{1}{z^2}\right)^{2n}=\frac{(-1)^n}{(2n)!}\cdot\frac{1}{z^{4n-1}}, \qquad \text{for } n=0,1,2,\ldots. \qquad \text{1A}$$

Since the Laurent series for g about 0 has infinitely many negative powers, g has an essential singularity at 0. $\frac{1}{2}$M, $\frac{1}{2}$A

(iii) Since g has an essential singularity at 0, we can apply the Casorati–Weierstrass Theorem. Take $\alpha=0$, $\delta=1$, and choose $\beta=1001i$ and $\varepsilon=1$, so that the disc with centre β and radius ε lies in 1M

$$\{w:\operatorname{Im} w>1000\}.$$

By the Casorati–Weierstrass Theorem, there exists z such that 1M

$$0<|z|<1 \quad\text{and}\quad |g(z)-\beta|<\varepsilon.$$

Hence $\operatorname{Im}(g(z))>1000$, as required. 1M

Question 11

(a) The function $\phi(z)=1/z^2$ is even and analytic on $\mathbb{C}$, apart from a pole at 0, and 1M

$$\begin{aligned} f(z) &= \frac{\pi\operatorname{cosec}\pi z}{z^2} \\ &= \frac{\pi}{z^2}\left(\frac{1}{\pi z}+\frac{\pi z}{6}+\frac{7}{360}(\pi z)^3+\cdots\right) && \text{1M} \\ &= \frac{1}{z^3}+\frac{\pi^2}{6}\frac{1}{z}+\cdots, \end{aligned}$$

so that $\operatorname{Res}(f,0)=\pi^2/6$. 1A

Also, if S_N is the square contour with vertices at $\left(N+\frac{1}{2}\right)(\pm1\pm i)$, then

$$|\operatorname{cosec}\pi z|\le 1, \qquad \text{for } z\in S_N, \qquad \text{1M}$$

and $|z|\ge N+\frac{1}{2}$, for $z\in S_N$, so that, by the Estimation Theorem,

$$\left|\int_{S_N} f(z)\,dz\right|\le\frac{\pi}{\left(N+\frac{1}{2}\right)^2}4(2N+1)=\frac{16\pi}{2N+1}, \qquad \tfrac{1}{2}\text{A}, \tfrac{1}{2}\text{M}$$

which tends to 0 as $N\to\infty$. Hence, by HB3, page 5, item 4.3 (*Unit C1*, Theorem 4.2), 1M

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\tfrac{1}{2}\operatorname{Res}(f,0)=-\frac{\pi^2}{12}. \qquad \text{1A}$$