

- (b) If Γ is any simple-closed contour, such as $\{z : |z| = \frac{1}{2}\}$, that surrounds 0 but not 1, then

$$\int_{\Gamma} f(z) dz = \int_{\Gamma} \frac{h(z)}{z} dz,$$

where the function $h(z) = (\exp z)/(z-1)^3$ is analytic on a simply-connected region containing Γ . Hence, by Cauchy's Integral Formula, for such a Γ ,

$$\int_{\Gamma} f(z) dz = 2\pi i h(0) = -2\pi i.$$

- (c) The function f is analytic on \mathbb{C} except for the two singularities at 0, 1 and each C_r , for $r > 1$, is a simple-closed contour surrounding both of these singularities. Hence, by the Residue Theorem, for each $r > 1$,

$$\phi(r) = \int_{C_r} f(z) dz = 2\pi i (\text{Res}(f, 0) + \text{Res}(f, 1)).$$

Thus ϕ is a constant function.

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Question 10

- (a) (i) The function f is analytic everywhere except at 2 and -2 . Hence it is analytic on the punctured discs

$$D_1 = \{z : 0 < |z-2| < 2\} \text{ and } D_2 = \{z : 0 < |z+2| < 2\},$$

and so the only singularities of f are at 2 and -2 .

$\frac{1}{2}$ A, $\frac{1}{2}$ M

Since

$$f(z) = \frac{g_1(z)}{z-2}, \quad \text{for } z \in D_1,$$

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where the function $g_1(z) = 4/(z+2)$ is analytic on D_1 with $g_1(2) \neq 0$, f has a simple pole at 2.

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Similarly,

$$f(z) = \frac{g_2(z)}{z+2}, \quad \text{for } z \in D_2,$$

where the function $g_2(z) = 4/(z-2)$ is analytic on D_2 with $g_2(-2) \neq 0$, so f has a simple pole at -2 .

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- (ii) The function f has two Laurent series about the point 2, with annuli of convergence

$$\{z : 0 < |z-2| < 4\} \text{ and } \{z : 4 < |z-2|\}.$$

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- (iii) Put $h = z-2$. Then

$$\begin{aligned} f(z) &= \frac{4}{z^2-4} \\ &= \frac{4}{(z-2)(z+2)} \\ &= \frac{4}{h(h+4)} \\ &= \frac{1}{h(1+h/4)} \\ &= \frac{1}{h} \left(1 - \frac{h}{4} + \left(\frac{h}{4}\right)^2 - \dots \right), \quad \text{for } 0 < |h| < 4, \\ &= \frac{1}{z-2} \left(1 - \frac{z-2}{4} + \left(\frac{z-2}{4}\right)^2 - \dots \right), \quad \text{for } 0 < |z-2| < 4. \end{aligned}$$

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Since $\{z : 0 < |z-2| < 1\} \subseteq \{z : 0 < |z-2| < 4\}$, this is the Laurent series about 2 on $\{z : 0 < |z-2| < 1\}$.

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The general term of this series is of the form

$$\frac{(-1)^n}{z-2} \left(\frac{z-2}{4}\right)^n = \frac{(-1)^n}{4^n} (z-2)^{n-1}, \quad \text{for } n = 0, 1, 2, \dots$$

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