Question 11

(a) Given that the Laurent series about 0 for the function cosec is

$$\operatorname{cosec} z = \frac{1}{z} + \tfrac{1}{6}z + \tfrac{7}{360}z^3 + \cdots,$$

determine the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. [7]

(b) Let $D = \{z : |z| < 1\}$ and let f be the function

$$f(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \qquad (z \in D).$$

(i) Use Weierstrass' Theorem to prove that f is analytic, and hence obtain a formula for the derivative of f.

(ii) Determine an analytic function g with domain $\mathbb{C} - \{x \in \mathbb{R} : x \geq 1\}$ which is a direct analytic continuation of f.

(iii) Write down an analytic function h which is an indirect analytic continuation of f. [11]

Question 12

(a) State whether each of the following assertions is true or false, *briefly* justifying your answers.

(i) All analytic functions are conformal mappings.

(ii) All generalized circles lie in $\mathbb{C}$.

(iii) All linear functions are Möbius transformations. [6]

(b) (i) Determine the image of $\mathcal{R} = \{z : |z| < 1, \operatorname{Re} z < 0\}$ under the mapping given by

$$z_1 = \frac{z+i}{-z+i}.$$

(ii) Determine the image of $\mathcal{R}_1 = \{z_1 : \operatorname{Re} z_1 > 0, \operatorname{Im} z_1 > 0\}$ under the mapping given by

$$z_2 = z_1^2.$$

(iii) Determine a Möbius transformation which maps $\mathcal{R}_2 = \{z_2 : \operatorname{Im} z_2 > 0\}$ onto the open unit disc S.

(iv) Hence obtain a formula for a one-one conformal mapping f from $\mathcal{R}$ onto S and a formula for the corresponding inverse function f^{-1}. (In both cases, you need not simplify your answer.) [12]

[END OF QUESTION PAPER]