(b) (i) To apply Weierstrass' Theorem, we need to show that the series defining f is uniformly convergent on each closed disc in $D = \{z : |z| < 1\}$.

Let $E = \{z : |z| \le r\}$, where $0 < r < 1$, and put $\phi_n(z) = z^n/n$, for $n = 1, 2, 3, \ldots$. Then — 1M for approach using M-test

$$|\phi_n(z)| \le \frac{r^n}{n} \le r^n, \qquad \text{for } n = 1, 2, 3, \ldots,$$

so that Assumption 1 of the M-test holds with $M_n = r^n$. Also — 1M for Assumption 1

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n \text{ is convergent (since } r < 1), \qquad \text{1M for Assumption 2}$$

and so Assumption 2 also holds. Thus, by the M-test, $\sum_{n=1}^{\infty} \phi_n(z)$ is uniformly convergent on E, and so on each closed disc in $\{z : |z| < 1\}$. — 1M for M-test

Therefore, by Weierstrass' Theorem, f is analytic on $\{z : |z| < 1\}$, and its derivative can be obtained from term by term differentiation: — 1M for correct application

$$f'(z) = \sum_{n=1}^{\infty} \frac{nz^{n-1}}{n} = \sum_{n=1}^{\infty} z^{n-1} = 1 + z + z^2 + \cdots. \qquad \text{1A}$$

(ii) The function $g(z) = -\operatorname{Log}(1 - z)$ is analytic on $\mathbb{C} - \{x \in \mathbb{R} : x \ge 1\}$ and its Taylor series about 0 is — 1A

$$g(z) = -\left((-z) - \frac{(-z)^2}{2} + \frac{(-z)^3}{3} - \cdots\right) \qquad \text{1M, 1A}$$

$$= z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots, \qquad \text{for } |z| < 1.$$

Therefore g agrees with f on $\{z : |z| < 1\}$, and so g is a direct analytic continuation of f. — 1M

(iii) The function

$$h(z) = -\operatorname{Log}(1 - z) \qquad (|z + 3| < 1) \qquad \text{1A}$$

is a direct analytic continuation of g, but not of f, so it is an indirect analytic continuation of f.

Question 12

(a) (i) False. The analytic function $f(z) = z^2$ is not conformal at 0, since $f'(0) = 0$. — 1A, 1M

(ii) False. The extended line $\{z : \operatorname{Im} z = 0\} \cup \{\infty\}$ is a generalized circle, but it does not lie in $\mathbb{C}$. — 1A, 1M

(iii) True. A linear function is of the form — 1A, 1M

$$f(z) = az + b,$$

where $a \neq 0$, and this takes the form of a Möbius transformation:

$$f(z) = \frac{az + b}{cz + d},$$

where $c = 0$, $d = 1$ and $ad - bc = a \neq 0$.

(b) (i) The extended Möbius transformation corresponding to — 1M

$$z_1 = (z + i)/(-z + i),$$

maps $-i$ to 0, 0 to 1 and i to ∞. Also $\partial\mathcal{R}$ has a right-angled corner at $-i$ and so the image of $\partial\mathcal{R}$ has a right-angled corner at 0. Using the preservation of boundary orientation, we find that the image of $\mathcal{R}$ is $\mathcal{R}_1 = \{z_1 : \operatorname{Re} z_1 > 0, \operatorname{Im} z_1 > 0\}$. — 1M, 1M, $\frac{1}{2}$A

(ii) The mapping $z_2 = z_1^2$ squares the modulus and doubles the argument, so the image of $\mathcal{R}_1$ is — 1M

$$\mathcal{R}_2 = \{z_2 : \operatorname{Im} z_2 > 0\}. \qquad \tfrac{1}{2}\text{A}$$