Question 8

(a) We deduce from HB4, page 9, item 2.1 (*Unit D3*, Theorem 2.1), that 1M

$$z_{n+1} = z_n(1 - z_n) = -z_n^2 + z_n, \quad n = 0, 1, 2, \ldots,$$

is conjugate to

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \ldots,$$

where $d = -1 \times 0 + \frac{1}{2} \times 1 - \frac{1}{4} \times 1^2 = \frac{1}{4}$, with $\frac{1}{2}$A

$$w_0 = h(z_0) = -z_0 + \tfrac{1}{2} = 0,$$ $\frac{1}{2}$A

since $z_0 = \frac{1}{2}$.

(b) (i) The point $\frac{1}{2}i$ appears to lie inside the main cardioid (HB 4, page 10, figure). If $c = \frac{1}{2}i$, then $|c|^2 = \frac{1}{4}$ and $\operatorname{Re} c = 0$, so that

$$\left(8|c|^2 - \tfrac{3}{2}\right)^2 + 8 \operatorname{Re} c = \left(2 - \tfrac{3}{2}\right)^2 = \tfrac{1}{4} < 3.$$ 1A

Hence P_c has an attracting fixed point, and so $c \in M$, by HB4, page 10, item 4.8. 1M, 1A

(ii) The point $1 + \frac{1}{2}i$ appears to lie outside M (HB 4, page 10, figure). If $c = 1 + \frac{1}{2}i$, then the first few terms of $\{P_c^n(0)\}$ are:

$$1 + \tfrac{1}{2}i,$$

$$\left(1 + \tfrac{1}{2}i\right)^2 + \left(1 + \tfrac{1}{2}i\right) = 1 + i - \tfrac{1}{4} + 1 + \tfrac{1}{2}i$$ 1A

$$= \tfrac{7}{4} + \tfrac{3}{2}i.$$

Since $\left|\frac{7}{4} + \frac{3}{2}i\right| = \sqrt{\frac{85}{16}} > 2$, we deduce that $c \notin M$, by HB4, page 10, item 4.5. 1A, 1M

Solutions to Part II

Question 9

(a) (i) The function f is analytic on $\mathbb{C} - \{0, 1\}$, and so it is analytic on the simply-connected region $\{z : \operatorname{Re} z > 1\}$, which contains the given closed contour $\Gamma = \{z : |z - 2| = \frac{1}{2}\}$. Hence, for this Γ, by Cauchy's Theorem, 1M for details 1M for theorem

$$\int_\Gamma f(z)\, dz = 0.$$ 1A

(ii) The function $g(z) = (\exp z)/z$ is analytic on the simply-connected region $\{z : \operatorname{Re} z > 0\}$, which contains the given simple-closed contour $\Gamma = \{z : |z - 2| = \frac{3}{2}\}$. Hence, for this Γ, by Cauchy's nth Derivative Formula, with $n = 2$, 1M for approach 1M for details 1M for formula

$$\int_\Gamma f(z)\, dz = \int_\Gamma \frac{g(z)}{(z-1)^3}\, dz$$

$$= \frac{2\pi i}{2!} g^{(2)}(1) \qquad \text{(since 1 lies inside } \Gamma\text{)}.$$ 1A

Now

$$g'(z) = \frac{e^z(z-1)}{z^2},$$

and so

$$g''(z) = \frac{z^2(e^z + e^z(z-1)) - 2ze^z(z-1)}{z^4}$$

$$= \frac{(z^2 - 2z + 2)\, e^z}{z^3}.$$ 1A

Thus

$$\int_\Gamma f(z)\, dz = \frac{\pi i(1 - 2 + 2)e^1}{1} = e\pi i.$$ 1A