PART II

(i) You should attempt at most **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

Let f be the function defined by $f(z) = \dfrac{\exp z}{z(z-1)^3}$.

(a) Evaluate $\displaystyle\int_\Gamma f(z)\,dz$, where

(i) $\Gamma = \{z : |z-2| = \frac{1}{2}\}$,

(ii) $\Gamma = \{z : |z-2| = \frac{3}{2}\}$. [9]

(b) Find a simple-closed contour Γ such that

$$\int_\Gamma f(z)\,dz = -2\pi i.$$ [5]

(c) Let $\phi :]1, \infty[\longrightarrow \mathbb{C}$ be the function

$$\phi(r) = \int_{C_r} f(z)\,dz,$$

where $C_r = \{z : |z-1| = r\}$.

Show that ϕ is a constant function. (There is no need to evaluate the integral.) [4]

Question 10

(a) Let f be the function defined by $f(z) = 4/(z^2-4)$.

(i) Locate and classify the singularities of f.

(ii) How many different Laurent series does f have about 2?

What are the annuli of convergence of these Laurent series?

(iii) Determine the Laurent series about 2 for f, on the punctured open disc $\{z : 0 < |z-2| < 1\}$.

State an expression for the general term of this series. [*10*]

(b) Let g be the function defined by $g(z) = z\cos(1/z^2)$.

(i) Explain why the only singularity of g is at 0.

(ii) Determine the Laurent series about 0 for g, giving an expression for the general term of the series.

Hence classify the singularity of g at 0.

(iii) Prove that there is a complex number z such that $\operatorname{Im}(g(z)) > 1000$. [8]