Question 3

(a) The standard parametrization for the given line segment is

$$\begin{aligned}\gamma(t) &= (1-t)(-i)+ti\\ &= -i+2ti \qquad (t\in[0,1]).\end{aligned}$$ 1A

Thus

$$\begin{aligned}\int_\Gamma \operatorname{Im} z\,dz &= \int_0^1 \operatorname{Im}(\gamma(t))\gamma'(t)\,dt \qquad &\text{1M}\\ &= \int_0^1 (2t-1)2i\,dt\\ &= 2i\left[t^2-t\right]_0^1\\ &= 0. &\text{1A}\end{aligned}$$

(*Alternatively*, use the easier equivalent parametrization $\gamma(t)=it$ $(t\in[-1,1])$.)

(b) The length of the contour is $L=4\pi$. $\frac{1}{2}$A

By the backwards form of the Triangle Inequality, $\frac{1}{2}$M

$$\begin{aligned}\left|z^5-1\right| &\geq |z|^5-1\\ &= 2^5-1=31, \qquad \text{for } z\in C.\end{aligned}$$ $\frac{1}{2}$A

Also by the Triangle Inequality,

$$\begin{aligned}|\sinh z| &= \left|\frac{e^z-e^{-z}}{2}\right|\\ &\leq \tfrac{1}{2}\left(|e^z|+|e^{-z}|\right) &\text{1A, 1M}\\ &\leq e^{|z|} \qquad \left(\text{since } |e^z|\leq e^{|z|}\right)\\ &= e^2, \qquad \text{for } z\in C.\end{aligned}$$

Hence, by the Estimation Theorem (which applies since the integrand is continuous on C because $z^5-1\neq 0$ on C), 1M for theorem

$$\left|\int_C \frac{2\sinh z}{z^5-1}\,dz\right| \leq \frac{2e^2}{31}\times 4\pi = \frac{8\pi e^2}{31}.$$ $\frac{1}{2}$A

Question 4

(a) Let $z_n = z^n/(n-1)!$, $n=2,3,\ldots$. Then, for all $z\neq 0$,

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|z|}{n}\to 0 \qquad \text{as } n\to\infty.$$ 1A

Hence the radius of convergence is ∞, by the Ratio Test, and so the disc of convergence of the power series is $\mathbb{C}$. 1M for test 1A

(b) Since

$$\exp(-z) = 1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\cdots, \qquad \text{for } z\in\mathbb{C},$$ $\frac{1}{2}$A

and

$$\frac{1}{1-z} = 1+z+z^2+z^3+\cdots, \qquad \text{for } |z|<1,$$ $\frac{1}{2}$A

we deduce, by the Product Rule, that 1M for rule

$$\begin{aligned}f(z) &= \left(1-z+\frac{z^2}{2!}-\frac{z^3}{3!}+\cdots\right)\left(1+z+z^2+z^3+\cdots\right)\\ &= 1+(1-1)z+\left(1-1+\tfrac{1}{2!}\right)z^2+\left(1-1+\tfrac{1}{2!}-\tfrac{1}{3!}\right)z^3+\cdots &\text{1M for manipulation}\\ &= 1+\tfrac{1}{2}z^2+\tfrac{1}{3}z^3+\cdots, \qquad \text{for } |z|<1. &\text{1A for series}\end{aligned}$$

This Taylor series represents f on $\{z:|z|<1\}$. 1A for disc