Let $\Gamma_2 = \{z : |z| = 1\}$ and choose $g(z) = 7$. Then $f(z) - g(z) = z^7 + 5z^3$ and, for $z \in \Gamma_2$, — $\frac{1}{2}$A for g

$$\begin{aligned} |f(z) - g(z)| &= |z^7 + 5z^3| \\ &\leq |z|^7 + 5|z|^3 \qquad \text{(Triangle Inequality)} \\ &= 6, \end{aligned}$$ 1M

whereas

$$|g(z)| = 7 > 6, \qquad \text{for } z \in \Gamma_2.$$ $\frac{1}{2}$A

Hence, by Rouché's Theorem, f has the same number of zeros as g inside Γ_2, namely 0. — 1A

Also, f has no zeros on Γ_2, since $|f(z) - g(z)| < |g(z)|$, for $z \in \Gamma_2$, and so f has 7 zeros in $\{z : 1 < |z| < 2\}$. — 1M 1A

Question 7

(a) The function q is a model flow velocity function because the function $\overline{q}(z) = z^2$ is analytic (on $\mathbb{C}$). — 1M

(b) A complex potential function for q is

$$\Omega(z) = \tfrac{1}{3}z^3,$$ 1A

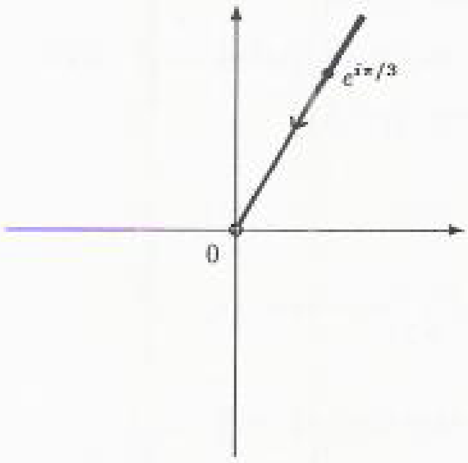
because this is a primitive of $\overline{q}$. The corresponding stream function is

$$\begin{aligned} \operatorname{Im}\Omega(z) &= \tfrac{1}{3}\operatorname{Im}(x+iy)^3 \qquad (z = x+iy) \\ &= \tfrac{1}{3}(3x^2y - y^3) \\ &= x^2y - \tfrac{1}{3}y^3. \end{aligned}$$ 1M, 1A

The streamline through the point $e^{i\pi/3} = \frac{1}{2}(1 + i\sqrt{3})$ satisfies

$$x^2y - \tfrac{1}{3}y^3 = \left(\tfrac{1}{2}\right)^2\sqrt{3}/2 - \tfrac{1}{3}(\sqrt{3}/2)^3 = 0.$$ $\frac{1}{2}$M

Since $x^2y - \frac{1}{3}y^3 = \frac{1}{3}y(3x^2 - y^2) = 0$ is satisfied on $y = 0$, $y = \sqrt{3}x$ and $y = -\sqrt{3}x$, and 0 is a stagnation point, the streamline is as shown below. — $\frac{1}{2}$M



1A ($\frac{1}{2}$A for streamline, $\frac{1}{2}$A for direction)

(c) Since q is a model flow velocity function on $\mathbb{C}$ it is locally flux-free; in particular, the flux across the unit circle is 0. — 1A, 1M