Question 5

(a) The function f has simple poles at $\pm 1/\sqrt{3}$ and $\pm\sqrt{3}$, of which only those at $\pm 1/\sqrt{3}$ lie inside the unit circle. By the g/h Rule (with $g(z) = z/(z^2-3)$, $h(z) = 3z^2 - 1$, $h'(z) = 6z$), we deduce that — 1A, 1M for rule

$$\begin{aligned}\operatorname{Res}(f, 1/\sqrt{3}) &= \frac{g(1/\sqrt{3})}{h'(1/\sqrt{3})} \\ &= \frac{1/\sqrt{3}}{(6/\sqrt{3})(1/3-3)} \\ &= -\tfrac{1}{16}\end{aligned}$$ $\tfrac{1}{2}$A

and

$$\begin{aligned}\operatorname{Res}(f, -1/\sqrt{3}) &= \frac{g(-1/\sqrt{3})}{h'(-1/\sqrt{3})} \\ &= \frac{-1/\sqrt{3}}{(-6/\sqrt{3})(1/3-3)} \\ &= -\tfrac{1}{16}.\end{aligned}$$ $\tfrac{1}{2}$A

(b) Using the strategy for trigonometric integrals, we have

$$\int_0^{2\pi} \frac{1}{1+3\sin^2 t}\,dt = \int_C \frac{1}{1+3((z-z^{-1})/2i)^2}\frac{1}{iz}\,dz$$ 1M

$$\begin{aligned}&= \int_C \frac{1}{iz\left(1-\frac{3}{4}(z^2-2+z^{-2})\right)}\,dz \\ &= -4i\int_C \frac{z}{z^2(4-3\,(z^2-2+z^{-2}))}\,dz \\ &= 4i\int_C \frac{z}{3z^4-10z^2+3}\,dz \\ &= 4i\int_C \frac{z}{(3z^2-1)(z^2-3)}\,dz,\end{aligned}$$ 2A for manipulation

where C is the unit circle.

Thus, by part (a) and the Residue Theorem, — 1M for theorem

$$\begin{aligned}\int_0^{2\pi} \frac{1}{1+3\sin^2 t}\,dt &= 4i\cdot 2\pi i\left(\operatorname{Res}(f, 1/\sqrt{3}) + \operatorname{Res}(f, -1/\sqrt{3})\right) \\ &= -8\pi(-\tfrac{1}{16} - \tfrac{1}{16}) \\ &= \pi.\end{aligned}$$ 1A

Question 6

Let $\Gamma_1 = \{z : |z| = 2\}$, $f(z) = z^7 + 5z^3 + 7$ and choose $g(z) = z^7$. — $\tfrac{1}{2}$A for g

Then $f(z) - g(z) = 5z^3 + 7$ and, for $z \in \Gamma_1$,

$$\begin{aligned}|f(z) - g(z)| &= |5z^3 + 7| \\ &\le 5|z|^3 + 7 \qquad \text{(Triangle Inequality)} \\ &= 47,\end{aligned}$$ 1M

whereas

$$|g(z)| = |z|^7 = 128 > 47, \qquad \text{for } z \in \Gamma_1.$$ $\tfrac{1}{2}$A

Hence, by Rouché's Theorem, f has the same number of zeros as g inside Γ_1, namely 7. — 1A