Question 11

(a) Find the residues of the function

$$f(z) = \frac{\pi \cot \pi z}{9z^2 + 1}$$

at each of the points $0, \frac{1}{3}i, -\frac{1}{3}i$. [6]

(b) Hence determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{9n^2 + 1}.$$ [8]

(c) Use your result from part (b) to prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{9n^2 + 1} = \frac{\pi}{3} \coth \frac{\pi}{3}.$$ [4]

Question 12

(a) Determine the extended Möbius transformation $\widehat{f}_1$ which maps 0 to 0, ∞ to 1 and $1 + i$ to ∞. Hence calculate $\widehat{f}_1\left(\frac{1}{2}(1+i)\right)$. [3]

(b) Let $D_1 = \{z : |z - 1| < 1\}$, $D_2 = \{z : |z - i| < 1\}$, $R = D_1 \cap D_2$, $S = \{z_1 : 3\pi/4 < \operatorname{Arg}_{2\pi} z_1 < 5\pi/4\}$ and $T = \{w : \operatorname{Re} w > 0, \operatorname{Im} w > 0\}$.

(i) Sketch the regions R, S and T.

(ii) Show that $\widehat{f}_1(R) = S$.

(iii) Hence determine a conformal mapping f from R to T.

(iv) Obtain a formula for the inverse function of f. [15]

[END OF QUESTION PAPER]