PART II

(i) You should attempt at most **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) Let f be the function

$$f(z) = z(1 + \bar{z}).$$

(i) Write $f(x + iy)$ in the form $u(x, y) + iv(x, y)$, where u and v are real-valued functions.

(ii) Use the Cauchy-Riemann equations to show that f is differentiable at 0, but not analytic there.

(iii) Evaluate $f'(0)$. [8]

(b) Let g be the function $g(z) = z^3$.

(i) Show that g is conformal at i.

(ii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = ti \quad (t \in \mathbb{R}).$$

Show that Γ_1 and Γ_2 meet at the point i, and sketch Γ_1 and Γ_2 on the same diagram.

(iii) Describe the effect of g on a small disc centred at i and hence make a sketch showing the approximate directions of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(i)$. [10]

Question 10

Let f be the function

$$f(z) = \frac{z}{\sin z}.$$

(a) Show that the Laurent series about 0 for f is

$$\frac{z}{\sin z} = 1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots, \qquad \text{for } 0 < |z| < \pi.$$

Hence evaluate the integral

$$\int_C \frac{1}{z^2 \sin z}\, dz,$$

where C is the unit circle $\{z : |z| = 1\}$. [7]

(b) Write down the domain A of f. Use the Uniqueness Theorem to show that f is the only analytic function with domain A such that

$$f(iy) = \frac{y}{\sinh y}, \qquad \text{for } y > 0.$$ [5]

(c) Show that f has singularities at points of the form $k\pi$, $k \in \mathbb{Z}$, and classify these singularities. [6]