$1\ \frac{31}{33}\quad 2\ \frac{28\frac{1}{2}}{33}\quad 3\ \frac{27}{34}$

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1)a)i) $f(z) = \dfrac{z^2+1}{3z^3+10iz^2-3z}$

$= \dfrac{(z+i)(z-i)}{z(3z^2+10iz-3)}$

$= \dfrac{(z+i)(z-i)}{z(3z+i)(z+3i)}$

$= \dfrac{(z+i)(z-i)}{3z(z+i/3)(z+3i)}$

2A,M $f(z)$ has Simple poles at $z=0$, $-i/3$, $-3i$

Using the Cover-up Rule !

$\mathrm{Res}(f,0) = \dfrac{i \times -i}{3\times i/3 \times 3i} = \dfrac{-1}{3}$

$\mathrm{Res}(f,-i/3) = \dfrac{(-i/3+i)(-i/3-i)}{3\times -i/3\,(-i/3+3i)} = \dfrac{-8/9}{-8/3} = +\dfrac{1}{3}$

3A,M $\mathrm{Res}(f,-3i) = \dfrac{(-3i+i)(-3i-i)}{3\times-3i\,(-3i+i/3)} = \dfrac{-2\times-4}{-9\times-8/3} = \dfrac{1}{3}$

ii) $\displaystyle\int_0^{2\pi} \frac{\cos t}{5+3\sin t}\,dt$

Sub $z=e^{it}$ then $dz = iz\,dt \Rightarrow dt = \dfrac{dz}{iz}$

$\cos t = \dfrac{z+z^{-1}}{2}$ $\sin t = \dfrac{z+z^{-1}}{2i}$

And the integral from 0 to 2π becomes an integral around C, unit circle

$\displaystyle\int_C \frac{(z+z^{-1})/2}{5+3(z-z^{-1})/2i}\,\frac{dz}{iz}$

$\displaystyle\int \frac{(z^2+1)/2z}{5iz+3(z^2-1)/2}\,dz = \int \frac{z^2+1\,dz}{10iz^2+3z^3-3z}$

3A,M $\displaystyle= \int_C \frac{z^2+1}{3z^3+10iz^2-3z}\,dz$ $\displaystyle= \int_C f(z)\,dz$ with f as in part (i)

which is f from a) integrated around the unit circle. f has poles 0 and $-i/3$ inside C.

$\displaystyle\int \frac{z^2+1\,dz}{3z^3+10iz^2-3z} = 2\pi i(\mathrm{Res}(f,0)+\mathrm{Res}(f,-i/3))$

$= 2\pi i(-1/3+1/3) = 0.$

10 2A,M as required.