Question 3 (*Unit C3*) – 34 marks

(a) Let

$$f(z) = \sum_{n=0}^{\infty}(-1)^{n+1}(1-z)^n \quad (|z-1| < 1)$$

and

$$g(z) = -\sum_{n=0}^{\infty}\frac{(z-i)^n}{(2-i)^{n+1}} \quad (|z-i| < \sqrt{5}).$$

Prove that the functions f and g are direct analytic continuations of each other. [8]

(b) Use Theorem 1.1 to evaluate the improper integral

$$\int_0^{\infty}\frac{t^{2/3}}{t^2+1}\,dt.$$ [9]

(c) Prove that the series

$$\sum_{n=1}^{\infty}\frac{e^{z^n}}{n^3+1}$$

(i) converges uniformly on $\{z : |z| \le 1\}$; [6]

(ii) defines a function analytic on $\{z : |z| < 1\}$. [3]

(d) (i) Evaluate $\Gamma(2i+3)/\Gamma(2i-1)$. [3]

(ii) Show that $\int_C \Gamma(z)\,dz = \pi i$, where $C = \{z : |z-i| = 3\}$. [5]

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Question 1 (*Unit D1*) – 34 marks

(a) Let f be the Möbius transformation

$$f(z) = \frac{2z+2}{z+i},$$

and let $C = \{z : |z+2-i| = \sqrt{5}\}$. Determine the image $f(C)$:

(i) in Apollonian form; [7]

(ii) in the form of Equation (1.1) of Theorem 1.2 on page 9. [5]

(b) Determine the point α such that α and $\beta = 3+2i$ are inverse points with respect to the circle $C = \{z : |z-1-i| = \sqrt{2}\}$. [3]

(c) (i) Sketch each of the following regions:

$$\mathcal{R} = \{z : |z| < 1,\ 0 < \operatorname{Arg} z < \pi/4\},$$
$$\mathcal{R}_1 = \{z_1 : |z_1| < 1,\ \operatorname{Re} z_1 < 0\},$$
$$\mathcal{R}_2 = \{z_2 : \operatorname{Re} z_2 < 0,\ \operatorname{Im} z_2 > 0\},$$
$$\mathcal{S} = \{w : \operatorname{Im} w < 0\}.$$ [4]

(ii) Write down a one-one conformal mapping from $\mathcal{R}$ onto $\mathcal{R}_1$. [2]

(iii) Determine a Möbius transformation which maps $\mathcal{R}_1$ onto $\mathcal{R}_2$. [6]

(iv) Write down a one-one conformal mapping from $\mathcal{R}_2$ onto $\mathcal{S}$. [1]

(v) Hence obtain a one-one conformal mapping f from $\mathcal{R}$ onto $\mathcal{S}$. [3]

(vi) Find the rule for the inverse mapping f^{-1} of the mapping f found in part (c)(v). [3]