(1)

First note that the result $\sum_{n=0}^{\infty} \omega^n = \frac{1}{1-\omega}$ is only valid for $|\omega|<1$.

$\therefore\ f(z) = \sum_{n=0}^{\infty}(-1)^{n+1}(1-z)^n = -\sum_{n=0}^{\infty}(z-1)^n = -\frac{1}{1-(z-1)}$ for $|z-1|<1$

i.e. $f(z) = \frac{-1}{2-z} = \frac{1}{z-2}$ for $z \in R$

Similarly

$g(z) = -\frac{1}{2-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{2-i}\right)^n = \frac{-1}{2-i}\cdot\frac{1}{1-\left(\frac{z-i}{2-i}\right)}$ for $\left|\frac{z-i}{2-i}\right|<1$

i.e. $g(z) = \frac{1}{z-2}$ for $|z-i|<|2-i|=\sqrt{5}$

i.e. $g(z) = \frac{1}{z-2}$ for $z \in S$

(2) Hence $f(z) = g(z)$ for $z \in T = R \cap S$

Since $R \cap S$ is non-empty, $T$ is a region with $T \subset R \cap S$
$\therefore$ $f$ and $g$ are direct analytic continuations.

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(3) As you are not attempting a solution from first principles here you have clearly(?) used Th^m 1.1 (C3 pg 13). In doing this you should have checked the conditions and set out your solution in terms of $f_1$ and $f_2$ of this theorem. i.e.

With $p(z)=1$, $q(z)=z^2+1$ and $a=2/3$ the conditions of Th^m 1.1 hold.

$f_1(z) = \frac{1}{z^2+1}\exp\left(\frac{2}{3}\operatorname{Log}_{2\pi} z\right)$ in $\mathbb{C}_{2\pi}$ are simple poles at $\pm i$.

$f_2(z) = \frac{1}{z^2+1}\exp\left(\frac{2}{3}\operatorname{Log} z\right)$ has no singularities on the +ve real axis.

So by Th^m 1.1 $\int_0^{\infty} = -\pi e^{-2\pi i/3}\operatorname{cosec}\left(\frac{2\pi}{3}\right) S - \pi\cot\left(2\pi/3\right) T$

with $S = \operatorname{Res}(f_1, i) + \operatorname{Res}(f_1, -i)$ and $T=0$

You have then found $S$ correctly and hence the required result!