

Question 11

- (a) (i) Use Rouché's Theorem to show that the equation

$$z^3 + 3z^2 + 1 = 0$$

has two solutions in the disc $\{z : |z| < 1\}$ and one in the annulus $\{z : 1 < |z| < 4\}$.

- (ii) Explain why the solution in the annulus is in fact real and negative. [8]

- (b) (i) Show that

$$\max\{|e^{z^3}| : |z| \leq 3\} = e^{27}$$

and find the point, or points, at which this maximum is attained.

- (ii) Hence find an upper estimate for

$$\left| \int_C \frac{z+1}{z-1} e^{z^3} dz \right|$$

where C is the circle $\{z : |z| = 3\}$.

[10]

Question 12

- (a) Determine the Möbius transformation which maps the points $1, 2i, \infty$ to the points $i, \infty, -1$, respectively. [5]

- (b) (i) Sketch the region

$$\mathcal{R} = \{z : |z| < 1, \operatorname{Im} z > 1 - \operatorname{Re} z\}.$$

- (ii) Determine and sketch the image of \mathcal{R} under the Möbius transformation

$$f_1(z) = \frac{z-i}{z-1}.$$

- (iii) Hence determine a one-one conformal mapping which maps \mathcal{R} onto the open upper half-plane. [13]

[END OF QUESTION PAPER]