PART II

(i) You should attempt at most **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) Use the Cauchy-Riemann Theorem and its converse to determine the point, or points, of $\mathbb{C}$ at which the function

$$f(z) = \overline{z}(1 - z)$$

is differentiable. *[8]*

(b) Let g be the function $g(z) = \dfrac{1}{z^2}$.

(i) Show that g is conformal at i.

(ii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = (1 - t)i - t \quad (t \in \mathbb{R}).$$

Show that Γ_1 and Γ_2 meet at the point i, and sketch Γ_1 and Γ_2 on the same diagram.

(iii) Describe the effect of g on a small disc centred at i and hence make a sketch showing the approximate directions of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(i)$. *[10]*

Question 10

(a) Let f be the function $f(z) = \dfrac{1}{z(z-3)}$.

(i) Locate and classify the singularities of f.

(ii) Find the Laurent series about 1 for f on the annulus $\{z : 1 < |z - 1| < 2\}$, giving the constant term and two terms on each side of it. *[9]*

(b) (i) Find the Taylor series about 0 (up to the term in z^4) for the function

$$g(z) = \exp(\cos z - 1),$$

and explain why the series represents g on $\mathbb{C}$.

(ii) Hence evaluate the integral

$$\int_C z^3 g(1/z)\, dz,$$

where C is the circle $\{z : |z| = 2\}$. *[9]*