PART II

(i) You should attempt at most **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) Use the Cauchy-Riemann Theorem and its converse to determine all the points of $\mathbb{C}$ at which the function

$$f(z) = \sin \overline{z}$$

is differentiable. [8]

(b) Let g be the function $g(z) = z^2$.

(i) Show that g is conformal at i.

(ii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = (t-1) + it \quad (t \in \mathbb{R}).$$

Show that Γ_1 and Γ_2 meet at the point i, and sketch Γ_1 and Γ_2 on the same diagram.

(iii) Describe the effect of g on a small disc centred at i and hence make a sketch showing the approximate directions of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(i)$. [10]

Question 10

(a) Let f be the function

$$f(z) = \frac{\sin z}{z(z-2)^3}.$$

Write down the singularities of f and determine their nature. [5]

(b) (i) Write down the Laurent series about 0 for the function

$$g(z) = \sin(1/z),$$

giving an expression for the general term of the series, and state its annulus of convergence. [3]

(ii) Hence evaluate the integral

$$\int_C z^6 \sin(1/z)\, dz,$$

where C is the unit circle $\{z : |z| = 1\}$. [3]

(c) Determine the first three non-zero terms of the Taylor series about 0 for the function

$$h(z) = \operatorname{Log}(\cos z),$$

and hence determine the first three non-zero terms of the Taylor series for tan. [7]