Question 11

(a) Find the residues of the function

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{9z^2 + 1}$$

at each of the points $0, \frac{1}{3}i, -\frac{1}{3}i$. [6]

(b) Hence determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{9n^2 + 1}.$$ [8]

(c) Generalize the method in parts (a) and (b) to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{k^2 n^2 + 1}, \quad \text{where } k = 1, 2, 3, \ldots.$$ [4]

Question 12

(a) Determine the extended Möbius transformation $\hat{f}_1$ which maps i to 0, 1 to ∞ and $\frac{1}{2}(1+i)$ to 1. [4]

(b) Let $R = \{z : |z| < 1, \operatorname{Re} z + \operatorname{Im} z > 1\}$ and $S = \{w : \operatorname{Re} w > 0, \operatorname{Im} w > 0\}$.

(i) Sketch the regions R and S.

(ii) Determine the image of R under $\hat{f}_1$ of part (a).

(iii) Hence determine a conformal mapping f from R onto S.

(iv) Write down the rule of the inverse function f^{-1}. [14]

[END OF QUESTION PAPER]