

PART II

- (i) You should attempt at most **TWO** questions in this part.
 (ii) Each question in this part carries 18 marks.

Question 9

- (a) Determine for which values of the real constants a and b the function

$$f(z) = z^2 + a(\operatorname{Re} z)^2 + ib(\operatorname{Im} z)^2$$

is analytic on \mathbb{C} .

[8]

- (b) Let g be the function

$$g(z) = z^2 - 2.$$

- (i) Determine the set of points at which g is conformal.

- (ii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = 1 + 2e^{it} \quad (t \in [0, 2\pi))$$

$$\Gamma_2 : \gamma_2(t) = it \quad (t \in \mathbb{R}).$$

Sketch Γ_1 and Γ_2 . On a separate diagram sketch the approximate shape of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(i\sqrt{3})$, showing clearly the directions of these paths at $g(i\sqrt{3})$. Briefly justify your sketch.

[10]

Question 10

Let f be the function

$$f(z) = \exp\left(\frac{1}{1-z}\right),$$

and let $C_1 = \{z : |z| = \frac{1}{2}\}$, $C_2 = \{z : |z| = 2\}$.

- (a) Show that f has just one singularity, and determine its nature.

[3]

- (b) Evaluate the following integrals.

(i) $\int_{C_1} f(z) dz$

(ii) $\int_{C_1} \frac{f(z)}{(4z-1)^2} dz$

(iii) $\int_{C_2} f(z) dz$

(iv) $\int_{C_2} \frac{f(z)}{z} dz$

[15]