PART II

(i) You should attempt at most **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) (i) Prove that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \qquad \text{for } z = x + iy \in \mathbb{C},$$

and obtain a similar formula for $|\cos z|^2$.

(ii) Deduce that

$$|\tan z|^2 \le 1, \qquad \text{for } -\tfrac{1}{4}\pi \le \operatorname{Re} z \le \tfrac{1}{4}\pi.$$ [10]

(b) Determine for which values of the real numbers a and b the function

$$f(z) = x^2 + 2axyi + by^2, \qquad \text{where } z = x + iy,$$

is analytic on $\mathbb{C}$. [8]

Question 10

Let f be the function $f(z) = \dfrac{\sin z}{z(z^2+9)}$.

(a) Find the residues of f at each of the singularities of f. [4]

(b) Evaluate $\int_\Gamma f(z)\,dz$, where

(i) $\Gamma = \{z : |z| = 1\}$;

(ii) $\Gamma = \{z : |z| = 4\}$. [5]

(c) Find a simple-closed contour Γ such that

$$\int_\Gamma f(z)\,dz = \frac{\pi \sinh 3}{9}.$$ [4]

(d) Evaluate

$$\int_\Gamma \frac{1}{z f(z)}\,dz,$$

where $\Gamma = \{z : |z| = 2\}$. [5]

Question 11

(a) (i) Show that

$$\left|\exp\left(e^{-z}\right)\right| = \exp\left(e^{-x}\cos y\right), \qquad \text{for } z = x + iy \in \mathbb{C}.$$

(ii) Determine

$$\max\left\{\left|\exp\left(e^{-z}\right)\right| : -1 \le \operatorname{Re} z \le 1,\ -\pi \le \operatorname{Im} z \le \pi\right\},$$

and find the point or points at which this maximum is attained. [10]

(b) Let $r > 0$. Show that the functions

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{r}\right)^n \qquad (|z| < r)$$

and

$$g(z) = -\sum_{n=1}^{\infty} \left(\frac{r}{z}\right)^n \qquad (|z| > r)$$

are indirect analytic continuations of each other. [8]