Question 5

(a) Find the residues of the function

$$f(z) = \frac{e^{3iz}}{z^2 + 4}$$

at each of the poles of f. [4]

(b) Hence evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3t}{t^2 + 4}\, dt$. [4]

Question 6

(a) Show that

$$|e^z| \geq \tfrac{1}{3}, \qquad \text{for } |z| = 1.$$ [1]

(b) Use Rouché's Theorem and the result of part (a) to determine the number of zeros of the function

$$f(z) = e^z - \tfrac{1}{3}z^4$$

in the open disc $\{z : |z| < 1\}$. [5]

(c) Hence evaluate $\displaystyle\int_\Gamma \frac{1}{f(z)}\, dz$, where $\Gamma = \{z : |z| = \tfrac{1}{2}\}$. [2]

Question 7

Let $q(z) = \dfrac{2}{\bar{z}}$ be a velocity function.

(a) Explain why q represents a model fluid flow. [1]

(b) Determine a complex potential function for this flow. Hence sketch the streamline through the point i and the streamline through the point $1 + i$. In each case indicate the direction of flow. [5]

(c) Evaluate the flux of q across the unit circle $\{z : |z| = 1\}$. [2]

Question 8

(a) Prove that the iteration sequence

$$z_{n+1} = 2z_n(1 - z_n), \qquad n = 0, 1, 2, \ldots,$$

with $z_0 = -1$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2, \qquad n = 0, 1, 2, \ldots,$$

with $w_0 = 3$. [2]

(b) Which of the following points c lies in the Mandelbrot set.

(i) $c = -1 - i$

(ii) $c = -\frac{1}{4}i$

Justify your answer in each case. [6]