M337/P

Third Level Course Examination 1999

Complex Analysis

Tuesday, 19 October, 1999 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I (64% of the marks) you should attempt as many questions as you can.

In Part II (36% of the marks) you should attempt no more than **TWO** questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach all your answer books together using the fastener provided.

The use of calculators is **NOT** permitted in this examination.

PART I

(i) You should attempt as many questions as you can in this part.

(ii) Each question in this part carries 8 marks.

Question 1

Determine each of the following complex numbers in Cartesian form, simplifying your answers as far as possible.

(a) $(1+i)^4$ [2]

(b) $\cos(\pi - i\log_e 2)$ [3]

(c) $(-e)^{i\pi}$ [3]

Question 2

Let $A = \{z : 1 \leq |z - i| \leq 2\}$ and $B = \{z : -\pi/4 < \operatorname{Arg} z < \pi/4\}$.

(a) Make separate sketches of the sets A and B. [2]

(b) For each of the sets A, B and $C = \operatorname{ext} A$

(i) state whether or not it is a region, and if it is not a region, then explain why not;

(ii) state whether or not it is compact, and if it is not compact, then explain why not. [6]

Question 3

(a) Evaluate

$$\int_\Gamma \operatorname{Im} z \, dz,$$

where Γ is the line segment from i to 1. [4]

(b) Determine an upper estimate for the modulus of

$$\int_C \frac{\bar{z}^2 - 1}{z^2 - 1} \, dz,$$

where C is the circle $\{z : |z| = 2\}$. [4]

Question 4

(a) Evaluate the following integrals, where C is the circle with centre i and radius 2. Name any standard results that you use and check that their conditions hold.

(i) $$\int_C \frac{e^{i\pi z}}{z+1} \, dz$$

(ii) $$\int_C \frac{e^{i\pi z}}{z-2} \, dz$$ [6]

(b) Use Liouville's Theorem to establish that there is a complex number z such that

$$|\cos(1 - z^2)| > 100.$$ [2]

(a) Find the residues of the function

$$f(z) = \frac{z^2 + 1}{z(z - \frac{1}{2})(z - 2)}$$

at each of the poles of f. [4]

(b) Hence evaluate the integral

$$\int_0^{2\pi} \frac{\cos t}{5 - 4\cos t}\, dt.$$ [4]

Question 6

(a) Use Rouché's Theorem to show that the equation

$$2z^3 - 5z - 1 = 0$$

has three solutions inside the circle $C_1 = \{z : |z| = 2\}$, exactly one of which lies inside the circle $C_2 = \{z : |z| = 1\}$. [7]

(b) Show that the solution inside C_2 is real and positive. [1]

Question 7

Let $q(z) = i\bar{z}$ be a velocity function.

(a) Explain why q represents a model fluid flow on $\mathbb{C}$. [1]

(b) Determine a stream function for this flow. Hence find the equations of the streamlines through the points i and $1 + i$, and sketch these streamlines indicating the direction of flow. [6]

(c) Determine the flux of q across the path Γ, where

$$\Gamma : \gamma(t) = (1 + i)t \quad (t \in [1, 2]).$$ [1]

Question 8

(a) Show that i is an indifferent fixed point of the function

$$f(z) = z^2 - iz + i.$$ [3]

(b) Show that

(i) the point $1 - i$ does not lie in the Mandelbrot set;

(ii) the point $-\frac{9}{10} - \frac{\sqrt{3}}{10}i$ does lie in the Mandelbrot set. [5]

(i) You should attempt at most **TWO** questions in this part.
(ii) Each question in this part carries 18 marks.

Question 9

(a) Let f be the function

$$f(z) = z(1 + \bar{z}).$$

(i) Write $f(x + iy)$ in the form $u(x, y) + iv(x, y)$, where u and v are real-valued functions.

(ii) Use the Cauchy-Riemann equations to show that f is differentiable at 0, but not analytic there.

(iii) Evaluate $f'(0)$. [8]

(b) Let g be the function $g(z) = z^3$.

(i) Show that g is conformal at i.

(ii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = ti \quad (t \in \mathbb{R}).$$

Show that Γ_1 and Γ_2 meet at the point i, and sketch Γ_1 and Γ_2 on the same diagram.

(iii) Describe the effect of g on a small disc centred at i and hence make a sketch showing the approximate directions of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(i)$. [10]

Question 10

Let f be the function

$$f(z) = \frac{z}{\sin z}.$$

(a) Show that the Laurent series about 0 for f is

$$\frac{z}{\sin z} = 1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + \cdots, \qquad \text{for } 0 < |z| < \pi.$$

Hence evaluate the integral

$$\int_C \frac{1}{z^2 \sin z}\, dz,$$

where C is the unit circle $\{z : |z| = 1\}$. [7]

(b) Write down the domain A of f. Use the Uniqueness Theorem to show that f is the only analytic function with domain A such that

$$f(iy) = \frac{y}{\sinh y}, \qquad \text{for } y > 0.$$ [5]

(c) Show that f has singularities at points of the form $k\pi$, $k \in \mathbb{Z}$, and classify these singularities. [6]

(a) Find the residues of the function

$$f(z) = \frac{\pi \cot \pi z}{9z^2 + 1}$$

at each of the points $0, \frac{1}{3}i, -\frac{1}{3}i$. [6]

(b) Hence determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{9n^2 + 1}.$$ [8]

(c) Use your result from part (b) to prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{9n^2 + 1} = \frac{\pi}{3} \coth \frac{\pi}{3}.$$ [4]

Question 12

(a) Determine the extended Möbius transformation $\widehat{f}_1$ which maps 0 to 0, ∞ to 1 and $1 + i$ to ∞. Hence calculate $\widehat{f}_1\left(\frac{1}{2}(1 + i)\right)$. [3]

(b) Let $D_1 = \{z : |z - 1| < 1\}$, $D_2 = \{z : |z - i| < 1\}$, $R = D_1 \cap D_2$, $S = \{z_1 : 3\pi/4 < \text{Arg}_{2\pi}\, z_1 < 5\pi/4\}$ and $T = \{w : \text{Re}\, w > 0, \text{Im}\, w > 0\}$.

(i) Sketch the regions R, S and T.

(ii) Show that $\hat{f}_1(R) = S$.

(iii) Hence determine a conformal mapping f from R to T.

(iv) Obtain a formula for the inverse function of f. [15]

[END OF QUESTION PAPER]