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PART I

(i) You should attempt as many questions as you can in this part.

(ii) Each question in this part carries 8 marks.

Question 1

(a) Let $w = \dfrac{1}{1+i}$.

  (i) Determine $\operatorname{Arg} w$.

  (ii) Determine the principal fourth root of w in polar form. [5]

(b) Determine the Cartesian form of $(-1)^{3i}$, simplifying your answer as far as possible. [3]

Question 2

Let $A = \{z : 1 \le |z| \le 2\}$ and $B = \{z : 0 < \operatorname{Arg} z < \pi/2\}$.

(a) Make separate sketches of the sets A, B and $B - A$. [3]

(b) For each of the sets A, B and $B - A$

  (i) state whether or not it is a region, and if it is not a region, then explain why not;

  (ii) state whether or not it is compact, and if it is not compact, then explain why not. [5]

Question 3

Let Γ_1 be the line segment from 1 to i, and Γ_2 be the arc of the unit circle from 1 to i (anticlockwise). Evaluate the following integrals giving your answers in Cartesian form.

(a) $\displaystyle\int_{\Gamma_1} \operatorname{Re} z \, dz$ [4]

(b) $\displaystyle\int_{\Gamma_1} \frac{1}{z} \, dz$ [3]

(c) $\displaystyle\int_{\Gamma_2} \frac{1}{z} \, dz$ [1]

Question 4

Evaluate the following integrals naming any standard results that you use and checking that their required conditions hold.

(a) $\displaystyle\int_{C_1} \frac{z^3}{z^2+2} \, dz, \quad \text{where } C_1 = \{z : |z| = 1\}$ [2]

(b) $\displaystyle\int_{C_2} \frac{z^3}{z^2+2} \, dz, \quad \text{where } C_2 = \{z : |z| = 4\}$ [3]

(c) $\displaystyle\int_{C_2} \frac{z^3}{(z+2)^2} \, dz, \quad \text{where } C_2 = \{z : |z| = 4\}$ [3]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{1}{z^3 - 1}$$

at each of the poles of f. [4]

(b) Hence evaluate the real improper integral

$$\int_{-\infty}^{\infty} \frac{1}{t^3 - 1}\, dt.$$ [4]

Question 6

Let $D_0 = \{z : |z| < 2\}$ and $D_1 = \{z : |z| > 2\}$. Show that the analytic functions

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \qquad (z \in D_0)$$

and

$$g(z) = -\sum_{n=1}^{\infty} \left(\frac{2}{z}\right)^n \qquad (z \in D_1)$$

are indirect analytic continuations of each other. [8]

Question 7

Let $q(z) = 1/\overline{z}^2$ be a velocity function.

(a) Explain why q represents a model fluid flow on $\mathbb{C} - \{0\}$. [1]

(b) Determine a stream function for this flow. Hence find the equation of the streamline through the point i, and sketch this, indicating the direction of flow. [5]

(c) Find the flux of q across the unit circle. [2]

Question 8

(a) Find the fixed points of the function $f(z) = 2z(1 - z)$ and classify them as (super-)attracting, repelling or indifferent. [3]

(b) Which of the following points c lie in the Mandelbrot set:

(i) $c = -1 + i$;

(ii) $c = -1 - \frac{1}{8}i$?

Justify your answer in each case. [5]

PART II

(i) You should attempt at most **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) Use the Cauchy-Riemann Theorem and its converse to determine the point, or points, of $\mathbb{C}$ at which the function

$$f(z) = \overline{z}(1 - z)$$

is differentiable. [8]

(b) Let g be the function $g(z) = \dfrac{1}{z^2}$.

(i) Show that g is conformal at i.

(ii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = (1 - t)i - t \quad (t \in \mathbb{R}).$$

Show that Γ_1 and Γ_2 meet at the point i, and sketch Γ_1 and Γ_2 on the same diagram.

(iii) Describe the effect of g on a small disc centred at i and hence make a sketch showing the approximate directions of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(i)$. [10]

Question 10

(a) Let f be the function $f(z) = \dfrac{1}{z(z-3)}$.

(i) Locate and classify the singularities of f.

(ii) Find the Laurent series about 1 for f on the annulus $\{z : 1 < |z - 1| < 2\}$, giving the constant term and two terms on each side of it. [9]

(b) (i) Find the Taylor series about 0 (up to the term in z^4) for the function

$$g(z) = \exp(\cos z - 1),$$

and explain why the series represents g on $\mathbb{C}$.

(ii) Hence evaluate the integral

$$\int_C z^3 g(1/z)\, dz,$$

where C is the circle $\{z : |z| = 2\}$. [9]

Question 11

(a) (i) Use Rouché's Theorem to show that the equation

$$z^3 + 3z^2 + 1 = 0$$

has two solutions in the disc $\{z : |z| < 1\}$ and one in the annulus $\{z : 1 < |z| < 4\}$.

(ii) Explain why the solution in the annulus is in fact real and negative. [8]

(b) (i) Show that

$$\max\{|e^{z^3}| : |z| \leq 3\} = e^{27}$$

and find the point, or points, at which this maximum is attained.

(ii) Hence find an upper estimate for

$$\left| \int_C \frac{\overline{z}+1}{\overline{z}-1} e^{z^3} \, dz \right|$$

where C is the circle $\{z : |z| = 3\}$. [10]

Question 12

(a) Determine the Möbius transformation which maps the points 1, $2i$, ∞ to the points $i, \infty, -1$, respectively. [5]

(b) (i) Sketch the region

$$\mathcal{R} = \{z : |z| < 1, \operatorname{Im} z > 1 - \operatorname{Re} z\}.$$

(ii) Determine and sketch the image of $\mathcal{R}$ under the Möbius transformation

$$f_1(z) = \frac{z-i}{z-1}.$$

(iii) Hence determine a one-one conformal mapping which maps $\mathcal{R}$ onto the open upper half-plane. [13]

[END OF QUESTION PAPER]