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PART I

(i) You should attempt as many questions as you can in this part.

(ii) Each question in this part carries 8 marks.

Question 1

Determine each of the following complex numbers in Cartesian form, simplifying your answers as far as possible.

(a) $\exp(1 + i\pi/6)$ [2]

(b) $\dfrac{1}{(1-i)^4}$ [2]

(c) $\operatorname{Log} i$ [2]

(d) $i^{(i/\pi)}$ [2]

Question 2

Let

$$A = \{z : 1 < |z - i| < 2\} \quad \text{and} \quad B = \{z : 0 \le \operatorname{Im} z \le 2\}.$$

(a) Make separate sketches of the sets A, B and $A - B$. [3]

(b) Write down which of the sets A, B and $A - B$, if any, is

(i) open;

(ii) a region;

(iii) closed;

(iv) compact. [5]

Question 3

(a) Determine the standard parametrization for the line segment Γ from 1 to i. [1]

(b) Evaluate

$$\int_\Gamma \operatorname{Re} z \, dz.$$ [2]

(c) Determine an upper estimate for the modulus of

$$\int_\Gamma \frac{\cosh(\operatorname{Re} z)}{4 + z^2} \, dz.$$ [5]

Question 4

Evaluate the following integrals, in which $C = \{z : |z| = 1\}$. Name any standard results that you use and check that their hypotheses are satisfied.

(a) $\displaystyle\int_C \frac{1}{z^3} \, dz$ [3]

(b) $\displaystyle\int_C \frac{\cos(z - \pi)}{z^3} \, dz$ [3]

(c) $\displaystyle\int_C \frac{\cos z}{(z - \pi)^3} \, dz$ [2]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{z^2+1}{z(2z+1)(z+2)}$$

at all its poles. [3]

(b) Hence evaluate the real integral

$$\int_0^{2\pi} \frac{2\cos t}{5+4\cos t}\,dt.$$ [5]

Question 6

Determine the number of zeros of the function

$$f(z) = z^3 + z - 3$$

in each of the following sets.

(a) $\{z : 1 \leq |z| < 2\}$ [6]

(b) $\{z : \operatorname{Im} z > 0\}$ [2]

Question 7

Let $q(z) = i\,\overline{z}$ be a velocity function.

(a) Explain why q represents a model fluid flow on $\mathbb{C}$. [1]

(b) Determine a stream function for this flow. Hence find the equations of the streamlines through the points 1 and i, and sketch these streamlines, indicating the direction of flow in each case. [6]

(c) Why is 0 neither a source nor a vortex? [1]

Question 8

(a) Prove that the iteration sequence

$$z_{n+1} = (z_n + 1)^2, \qquad n = 0, 1, 2, \ldots,$$

with $z_0 = -1$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + 1, \qquad n = 0, 1, 2, \ldots,$$

with $w_0 = 0$. [2]

(b) Find the fixed points of $P_1(z) = z^2 + 1$ and determine their nature. [3]

(c) Show that $1 \notin M$ and hence, or otherwise, determine whether or not 0 is in the keep set K_1 and deduce the behaviour of the sequence $\{P_1^n(0)\}$. [3]

PART II

(i) You should attempt at most **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) Use the Cauchy-Riemann Theorem and its converse to determine all the points of $\mathbb{C}$ at which the function

$$f(z) = \sin \overline{z}$$

is differentiable. [8]

(b) Let g be the function $g(z) = z^2$.

(i) Show that g is conformal at i.

(ii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = e^{it} \quad (t \in [0, 2\pi]),$$
$$\Gamma_2 : \gamma_2(t) = (t-1) + it \quad (t \in \mathbb{R}).$$

Show that Γ_1 and Γ_2 meet at the point i, and sketch Γ_1 and Γ_2 on the same diagram.

(iii) Describe the effect of g on a small disc centred at i and hence make a sketch showing the approximate directions of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(i)$. [10]

Question 10

(a) Let f be the function

$$f(z) = \frac{\sin z}{z(z-2)^3}.$$

Write down the singularities of f and determine their nature. [5]

(b) (i) Write down the Laurent series about 0 for the function

$$g(z) = \sin(1/z),$$

giving an expression for the general term of the series, and state its annulus of convergence. [3]

(ii) Hence evaluate the integral

$$\int_C z^6 \sin(1/z)\, dz,$$

where C is the unit circle $\{z : |z| = 1\}$. [3]

(c) Determine the first three non-zero terms of the Taylor series about 0 for the function

$$h(z) = \operatorname{Log}(\cos z),$$

and hence determine the first three non-zero terms of the Taylor series for tan. [7]

Question 11

This question involves improper integrals which you may assume exist.

(a) Give a brief reason why

$$\int_{-\infty}^{\infty} \frac{t}{t^4 - 1}\, dt = 0.$$ [1]

(b) Evaluate $\displaystyle\int_0^{\infty} \frac{1}{t^4 - 1}\, dt.$ [7]

(c) Evaluate $\displaystyle\int_0^{\infty} \frac{\sqrt{t}}{t^4 - 1}\, dt.$ [10]

Question 12

(a) Determine the extended Möbius transformation $\widehat{f_1}$ which maps i to 0, ∞ to 1 and $-i$ to ∞. [3]

(b) Let $R = \{z : |z - 1| < \sqrt{2}\} \cap \{|z + 1| < \sqrt{2}\}$, $S = \{z_1 : 3\pi/4 < \operatorname{Arg}_{2\pi} z_1 < 5\pi/4\}$ and $T = \{w : \operatorname{Re} w > 0\}$.

(i) Sketch the regions R, S and T.

(ii) Explain why $\widehat{f_1}(R) = S$.

(iii) Hence determine a one-one conformal mapping f from R to T.

(iv) Determine a one-one conformal mapping g from R to the open unit disc $D = \{z : |z| < 1\}$. [There is no need to simplify your answer.] [15]

[END OF QUESTION PAPER]