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PART I

(i) You should attempt as many questions as you can in this part.

(ii) Each question in this part carries 8 marks.

Question 1

(a) Determine each of the following complex numbers in Cartesian form, simplifying your answers as far as possible.

(i) $(1-i)^8$

(ii) $(1+i)^i$ [5]

(b) Show that

$$\tan 2i = \left(\frac{e^4-1}{e^4+1}\right) i.$$ [3]

Question 2

Let

$$A = \{z : 1 \le |z| \le 2\} \quad \text{and} \quad B = \{z : |\operatorname{Arg} z| < 3\pi/4\}.$$

(a) Make separate sketches of the sets A, B, $A - B$ and $A \cap B$. [4]

(b) For each of the sets $A - B$ and $A \cap B$, state whether the set is

(i) a region;

(ii) compact.

Give a brief reason in each case. [4]

Question 3

Let Γ be the line segment from 0 to $1+i$.

(a) Evaluate

$$\int_\Gamma \overline{z}^2 \, dz.$$ [4]

(b) Show that

$$\left| \int_\Gamma \exp(\overline{z}^2) \, dz \right| \le \sqrt{2}.$$ [4]

Question 4

Find the Laurent series for the function

$$f(z) = \frac{1}{z^2-1}$$

(a) about the point 1 on the punctured disc $\{z : 0 < |z-1| < 2\}$; [4]

(b) about the point 0 on the region $\{z : |z| > 1\}$. [4]

In each case, state the general term of the series.

Question 5

(a) Find the residues of the function

$$f(z) = \frac{1}{z^3 + 1}$$

at each of the poles of f. [4]

(b) Hence evaluate the real improper integral

$$\int_{-\infty}^{\infty} \frac{1}{t^3 + 1}\, dt.$$ [4]

Question 6

Use Rouché's Theorem to show that the equation

$$z^6 - 3iz^4 + 1 = 0$$

has exactly two solutions in the set $\{z : 1 < |z| < 2\}$. [8]

Question 7

Let $q(z) = \overline{z} + i$ be a velocity function.

(a) Explain why q represents a model fluid flow. [1]

(b) Determine a stream function for this flow and hence find equations for the streamline through the point 1 and the streamline through the point $-1 + i$. [4]

(c) Sketch the streamlines found in part (b), showing the direction of flow, and also indicate any degenerate streamlines. [3]

Question 8

(a) Prove that the function $f(z) = z^2 + \frac{1}{4}$ has exactly one fixed point, and that this fixed point is indifferent. [3]

(b) Determine which of the following points c lie in the Mandlebrot set.

(i) $c = -1 + i$.

(ii) $c = -\frac{1}{2} - \frac{1}{2}i$.

Justify your answer in each case. [5]

PART II

(i) You should attempt at most **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) Determine for which values of the real constants a and b the function

$$f(z) = z^2 + a(\operatorname{Re} z)^2 + ib(\operatorname{Im} z)^2$$

is analytic on $\mathbb{C}$. [8]

(b) Let g be the function

$$g(z) = z^2 - 2.$$

(i) Determine the set of points at which g is conformal.

(ii) Let Γ_1 and Γ_2 be the paths

$$\Gamma_1 : \gamma_1(t) = 1 + 2e^{it} \quad (t \in [0, 2\pi])$$
$$\Gamma_2 : \gamma_2(t) = it \quad (t \in \mathbb{R}).$$

Sketch Γ_1 and Γ_2. On a separate diagram sketch the approximate shape of the paths $g(\Gamma_1)$ and $g(\Gamma_2)$ near the point $g(i\sqrt{3})$, showing clearly the directions of these paths at $g(i\sqrt{3})$. Briefly justify your sketch. [10]

Question 10

Let f be the function

$$f(z) = \exp\left(\frac{1}{1-z}\right),$$

and let $C_1 = \{z : |z| = \frac{1}{2}\}$, $C_2 = \{z : |z| = 2\}$.

(a) Show that f has just one singularity, and determine its nature. [3]

(b) Evaluate the following integrals.

(i) $\displaystyle\int_{C_1} f(z)\,dz$

(ii) $\displaystyle\int_{C_1} \frac{f(z)}{(4z-1)^2}\,dz$

(iii) $\displaystyle\int_{C_2} f(z)\,dz$

(iv) $\displaystyle\int_{C_2} \frac{f(z)}{z}\,dz$ [15]

Question 11

(a) Find the residues of the function

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{9z^2 + 1}$$

at each of the points 0, $\frac{1}{3}i$, $-\frac{1}{3}i$. [6]

(b) Hence determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{9n^2 + 1}.$$ [8]

(c) Generalize the method in parts (a) and (b) to find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{k^2 n^2 + 1}, \quad \text{where } k = 1, 2, 3, \ldots.$$ [4]

Question 12

(a) Determine the extended Möbius transformation $\hat{f}_1$ which maps i to 0, 1 to ∞ and $\frac{1}{2}(1+i)$ to 1. [4]

(b) Let $R = \{z : |z| < 1, \operatorname{Re} z + \operatorname{Im} z > 1\}$ and $S = \{w : \operatorname{Re} w > 0, \operatorname{Im} w > 0\}$.

(i) Sketch the regions R and S.

(ii) Determine the image of R under $\hat{f}_1$ of part (a).

(iii) Hence determine a conformal mapping f from R onto S.

(iv) Write down the rule of the inverse function f^{-1}. [14]

[END OF QUESTION PAPER]