Question 1

(a) 2 marks

(a)(i) $|\alpha| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ (Unit A1, Section 2, Para. 2)

(a)(ii) Arg $\alpha = 3\pi/4$. (Unit A1, Section 2, Para. 8)

(b) 6 marks

(b)(i) $\alpha = 2\sqrt{2}\exp(3i\pi/4)$

$$\frac{1}{\alpha} = \frac{1}{2\sqrt{2}}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = -\frac{1}{4} - i\frac{1}{4}$$ (Unit A1, Section 2, Para. 12)

(b)(ii) The principal value of $\alpha^{1/3}$ is (Unit A1, Section 3, Para 4)

$$\left(2\sqrt{2}\right)^{1/3}\left(\cos\left(\frac{1}{3}\left(\frac{3\pi}{4}\right)\right) + i\sin\left(\frac{1}{3}\left(\frac{3\pi}{4}\right)\right)\right)$$

$$= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = 1 + i$$

(b)(iii) Log $\alpha = \log_e\left(2\sqrt{2}\right) + i(3\pi/4) = \frac{3}{2}\log_e 2 + \frac{3\pi}{4}i$ (Unit A2, Section 5, Para. 1)

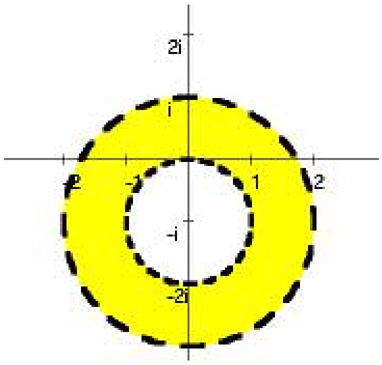
(b)(iv) $\text{Arg}(\alpha^3) = \frac{1}{4}\pi$ as $\frac{9}{4}\pi \equiv \frac{1}{4}\pi$.

Therefore $\text{Log}(\alpha^3) = 3\,\text{Log}\,\alpha - 2\pi = \frac{9}{2}\log_e 2 + \frac{\pi}{4}i$

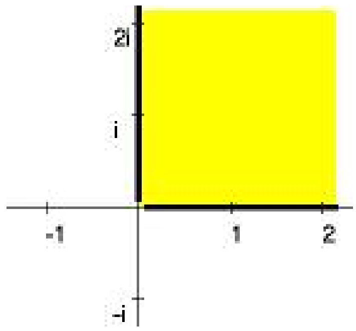
(Unit A2, Section 5, Paras. 1 & 2)

Question 2

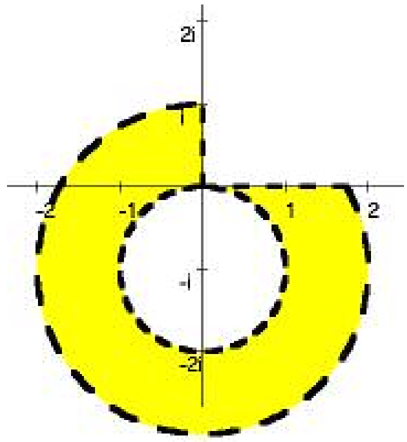
(a) 3 marks



A



B



C = A - B

Note origin not included in B as Arg not defined there. Also origin not in C.

(b) 4 marks

(b)(i)	A and C.
(b)(ii)	C.
(b)(iii)	B.

(c) 1 mark

$\{0, 1\}$.

Question 3

(a) 3 marks

(a)(i) The standard parametrization for the circle Γ is (Unit A2, Section 2, Para. 3)

$$\gamma(t) = 2(\cos t + i \sin t) = 2e^{it} \quad (t \in [0, 2\pi])$$

(a)(ii) $\gamma'(t) = 2ie^{it}$

Since γ is a smooth path then (Unit B1, Section 2, Para. 1)

$$\int_\Gamma \overline{z}\, dz = \int_0^{2\pi} \overline{\gamma(t)}\gamma'(t)dt = \int_0^{2\pi} 2e^{-it}\left(2ie^{it}\right)dt = 4i\int_0^{2\pi} dt = 8\pi i$$

(b) 5 marks

The length of Γ is $L = 2\pi * 2 = 4\pi$.

Using the Triangle Inequality (Unit A1, Section 5, Para. 3b) then, for $z \in \Gamma$, we have

$$\left|\overline{z}^2 - 1\right| \le \left|\overline{z}^2\right| + 1 = |z|^2 + 1 = 4 + 1 = 5$$

Using the Backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 3c) then, for $z \in \Gamma$, we have

$$\left|z^2 - 1\right| \ge \left|\,\left|z^2\right| - 1\,\right| = |4 - 1| = 3$$

Therefore $M = \left|\dfrac{\overline{z}^2 - 1}{z^2 - 1}\right| \le \dfrac{5}{3}$ for $z \in \Gamma$.

$f(z) = \dfrac{\overline{z}^2 - 1}{z^2 - 1}$ is continuous on $\mathbb{C} - \{-1, 1\}$ and hence on the circle Γ.

Therefore by the Estimation Theorem (Unit B1, Section 4, Para. 3)

$$\left|\int_\Gamma f(z)dz\right| \le ML = \frac{5}{3} * 4\pi = \frac{20}{3}\pi$$

Question 4

(a) 3 marks

$\mathbb{C}$ is a simply-connected region, C is a simple-closed contour in $\mathbb{C}$, and $f(z) = \exp(i\pi z)$ is analytic on $\mathbb{C}$.

As -1 lies inside the circle C then by Cauchy's Integral formula (Unit B2, Section 1, Para. 4) then

$$\int_C \frac{e^{i\pi z}}{z+1} dz = 2\pi i\, f(-1) = 2\pi i * e^{-i\pi} = -2\pi i$$

(b) 2 marks

Let $\boldsymbol{R} = \{z \in C : |z - i| < 5^{1/2}\}$. $\boldsymbol{R}$ is a simply-connected region and C is a simple-closed contour in $\boldsymbol{R}$. As $\frac{e^{i\pi z}}{z+3}$ is analytic on $\boldsymbol{R}$ then by Cauchy's Theorem (Unit B2, Section 1, Para. 4)

$$\int_C \frac{e^{i\pi z}}{z+3} dz = 0$$

(c) 3 marks Unit B2

Let $g(z) = \sin(z - \pi/2)$. g is a function which is analytic on the simply-connected region $\mathbb{C}$ (Unit B2, Section 1, Para. 3).

The contour C is a simple-closed contour in $\mathbb{C}$. Since z^3 is zero inside the circle C then using Cauchy's n^{th} Derivative Formula (Unit B2, Section 3, Para. 1), with n = 2 and $\alpha = 0$ we have

$$\int_C \frac{\sin(z-\pi/2)}{z^3} dz = \int_C \frac{g(z)}{z^3} dz = \frac{2\pi i}{2!} g^{(2)}(0)$$

$g'(z) = \cos(z - \pi/2)$.

$g''(z) = -\sin(z - \pi/2)$.

So $g''(0) = -\sin(-\pi/2) = 1$.

Hence $\int_C \frac{\sin(z-\pi/2)}{z^3} dz = \pi i$.

Question 5

(a) 3 marks

f is an analytic function with simple poles at z = 0, ½, and 2. Using the cover-up rule (Unit C1, Section 1, Para. 3).

$$\operatorname{Res}(f,0)=\frac{1}{\left(-\frac{1}{2}\right)(-2)}=1.$$

$$\operatorname{Res}\left(f,\tfrac{1}{2}\right)=\frac{\frac{1}{4}+1}{\frac{1}{2}\left(-\frac{3}{2}\right)}=-\frac{5}{3}.$$

$$\operatorname{Res}(f,2)=\frac{4+1}{2\left(\frac{3}{2}\right)}=\frac{5}{3}.$$

(b) 5 marks

I shall use the strategy given in Unit C1, Section 2, Para. 2.

$$\int_0^{2\pi}\frac{\cos t}{5-4\cos t}\,dt=\int_C\frac{\frac{1}{2}\left(z+z^{-1}\right)}{5-4\left(\frac{1}{2}\right)\left(z+z^{-1}\right)}\frac{1}{iz}\,dz$$, where C is the unit circle {z : |z| = 1}.

$$=-\frac{i}{2}\int_C\frac{z^2+1}{z\left(5z-2z^2-2\right)}dz$$

$$=\frac{i}{4}\int_C\frac{z^2+1}{z\left(z^2-\frac{5}{2}z+1\right)}dz=\frac{i}{4}\int_C\frac{z^2+1}{z\left(z-\frac{1}{2}\right)(z-2)}dz$$

f is analytic on the simply-connected region $\mathbb{C}$ except for a finite number of singularities. C is a simple contour in $\mathbb{C}$ not passing through any of the singularities. Since the singularities at z = ½, and 0 are inside the circle C then by Cauchy's Residue Theorem (Unit C1, Section 2, Para. 1) we have

$$\int_0^{2\pi}\frac{\cos t}{5-4\cos t}\,dt=\frac{i}{4}*2\pi i\left\{\operatorname{Res}\,(f,0)+\operatorname{Res}\left(f,\tfrac{1}{2}\right)\right\}$$

$$=-\frac{\pi}{2}\left\{1-\frac{5}{3}\right\}=\frac{\pi}{3}$$

Question 6

(a) 7 marks

(a)(i) Let $f(z) = 2z^3 + 5z - 1$ and $g_1(z) = 2z^3$.

For $z \in C_1$ then, using the Triangle Inequality (Unit A1, Section 5, Para. 3),

$$| f(z) - g_1(z) | = |5z - 1| \le |5z| + |-1| = 11 < 16 = | g_1(z) |.$$

As f is a polynomial then it is analytic on the simply-connected region $\boldsymbol{R} = \mathbb{C}$. Since C_1 is a simple-closed contour in $\boldsymbol{R}$ then by Rouché 's theorem (Unit C2, Section 2, Para. 4) f has the same number of zeros as g_1 inside the contour C_1. Therefore f has 3 zeros inside C_1.

(a)(ii) Let $g_2(z) = 5z$.
On the contour C_2 we have, using the Triangle Inequality,

$$| f(z) - g_2(z) | = |2z^3 - 1| \le |2z^3| + |-1| = 3$$
$$< 5 = | g_2(z) |.$$

As C_2 is a simple-closed contour in $\boldsymbol{R}$ then by Rouché's theorem f has the same number of zeros as g_2 inside the contour C_2. Therefore f has 1 zero inside C_2.

(b) 1 mark

$f(z) =0$ is a polynomial equation with real coefficients. Therefore if α is a solution then so is the complex conjugate $\overline{\alpha}$. If α is the only solution inside C_2 then we must have $\alpha = \overline{\alpha}$. Hence the solution is real.

Clearly α is non-zero. If $\alpha < 0$ then all the terms in $2\alpha^3 + 5\alpha - 1$ are negative so $\alpha < 0$ cannot be a solution. Therefore the solution inside C_2 is real and positive.

Question 7

(a) 1 mark

q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C}$ and the conjugate velocity function $\overline{q}(z) = -iz$ is analytic on $\mathbb{C}$**.** Therefore q is a model fluid flow on $\mathbb{C}$ (Unit D2, Section 1, Para. 14).

(b) 5 marks

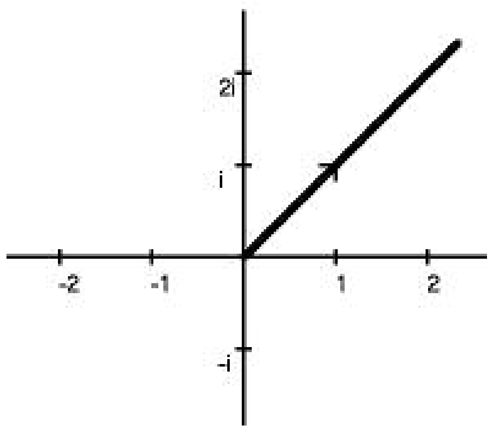
The complex potential function Ω is a primitive of $\overline{q}(z)$ (Unit D2, Section 2, Para. 1).
Therefore the complex potential function $\Omega(z) = -iz^2/2$ and the stream function

$$\Psi(x,y) = \operatorname{Im}\Omega(z) \quad \text{(Unit D2, Section 2, Para. 4)}$$

$$= \operatorname{Im}\left(-\tfrac{i}{2}(x+iy)^2\right) \quad \text{, where } z = x + iy$$

$$= \operatorname{Im}\left(-\tfrac{i}{2}\left(x^2 - y^2 + 2ixy\right)\right) = \tfrac{1}{2}\left(-x^2 + y^2\right)$$

A streamline through 1 + i is given by $\frac{1}{2}\left(-x^2 + y^2\right) = \Psi(1,1) = 0$.
Since the streamline goes through 1 + i it must have the equation y = x.
At 1 + i the velocity function q(1 + i) = i(1 - i) = 1 + i (north-east)



(c) 2 marks

Since Γ follows the streamline through 1 + i then the flux of q across Γ is 0 (Unit D2, Section 2, Para. 5).

Question 8

(a) 3 marks

Using the result in Unit D3, Section 2, Para. 1 then the iteration sequence $z_{n+1} = z_n^2 + 6z_n + 5$ is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + (1*5 + 6/2 - 6^2/4) = w_n^2 - 1$$

and conjugating function $h(z) = z + 3$.

Therefore $w_0 = h(z_0) = z_0 + 3 = -3 + 3 = 0$. (Unit D3, Section 1, Para. 7).

(b) 3 marks

If α is a fixed point of P_{-1} (Unit D3, Section 1, Para. 3) then

$$P_{-1}(\alpha) = \alpha^2 - 1 = \alpha.$$

The solutions of $\alpha^2 - \alpha - 1 = 0$ are $\dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$.

$P_{-1}{}'(z) = 2z$.

When $z = \dfrac{1 \pm \sqrt{5}}{2}$ then $|P_{-1}{}'(z)| = |1 \pm \sqrt{5}| > 1$.

Therefore $\dfrac{1 \pm \sqrt{5}}{2}$ are repelling fixed points (Unit D3 Section 1, Para. 5).

(c) 2 marks

Let $c = \frac{1}{2} - i$.

$P_c(0) = \frac{1}{2} - i$.

$P_c^2(0) = \left(\frac{1}{2} - i\right)^2 + \left(\frac{1}{2} - i\right) = \left(\frac{1}{4} - 1 - i\right) + \left(\frac{1}{2} - i\right) = -\frac{1}{4} - 2i$.

As $\left|P_c^2(0)\right| > 2$ then c does not lie in the Mandelbrot set (Unit D3, Section 4, Para. 5).

Question 9

(a) 8 marks

(a)(i)

$f(z) = \bar{z} + |z|^2 = (x - iy) + (x^2 + y^2) = u(x, y) + iv(x, y)$,
where u(x,y) = $x + x^2 + y^2$, and v(x,y) = -y.

(a)(ii)

$\frac{\partial u}{\partial x}(x, y) = 1 + 2x$, $\frac{\partial u}{\partial y}(x, y) = 2y$, $\frac{\partial v}{\partial x}(x, y) = 0$, $\frac{\partial v}{\partial y}(x, y) = -1$

If f is differentiable then the Cauchy-Riemann equations hold (Unit A4, Section 2, Para. 1). If they hold at (a, b)

$$\frac{\partial u}{\partial x}(a, b) = 1 + 2a = -1 = \frac{\partial v}{\partial y}(a, b), \text{ and}$$
$$\frac{\partial v}{\partial x}(a, b) = 0 = -2b = -\frac{\partial u}{\partial y}(a, b)$$

Therefore the Cauchy-Riemann equations only hold at (-1, 0).

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous at (-1, 0).
3. satisfy the Cauchy-Riemann equations at (-1, 0)

then, by the Cauchy-Riemann Converse Theorem (Unit A4, Section 2, Para. 3), f is differentiable at -1.

As the Cauchy-Riemann only hold at (-1, 0) then f is not differentiable on any region surrounding 0. Therefore f is not analytic at -1. (Unit A4, Section 1, Para. 3)

(a)(iii)

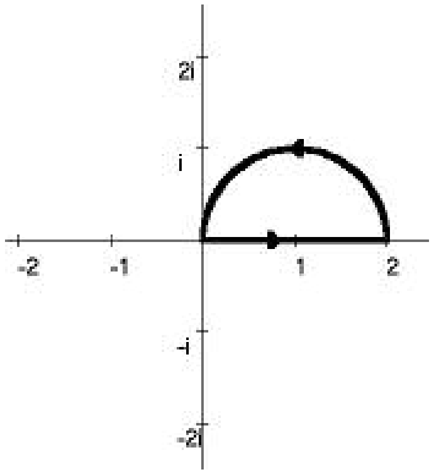
$f'(-1, 0) = \frac{\partial u}{\partial x}(-1,0) + i\frac{\partial v}{\partial x}(-1,0) = -1$ (Unit A4, Section 2, Para. 3).

(b) 10 marks

(i) The domain of g is ℂ (Unit A4, Section 1, Para. 7) and its derivative $g'(z)=3iz^2$ also has domain ℂ (Unit A4, Section 3, Para. 4) . Therefore g is analytic on ℂ - {0}. Since $g'(z) \neq 0$ on ℂ - {0}then g is conformal on ℂ - {0} (Unit A4, Section 4, Para. 6).

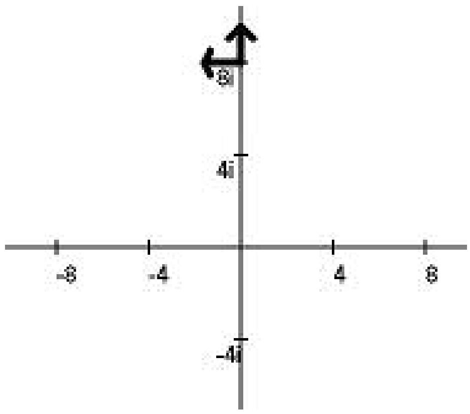
(ii) As g is analytic on ℂ and $g'(2) \neq 0$ then a small disc centred at 2 is mapped approximately (Unit A4, Section 1, Para. 11) to a small disc centred at $g(2) = 8i$. The disc is rotated by Arg $(g'(2))$ = Arg $12i = \pi/2$, and scaled by a factor $|g'(2)| = 12$.

(iii)



(iv)

The vertical line in the diagram below is $g(\Gamma_1)$. (Unit A4, Section 4, Para. 4)



(v) $(g_o\gamma_1)'(t) = g'(\gamma_1(t))\, \gamma_1'(t) = 3i(2t)^2\, 2 = 24it^2$.
$(g_o\gamma_2)'(t) = g'(\gamma_2(t))\, \gamma_2'(t) = 3i(1+e^{it})^2\, ie^{it}$.

Since $\gamma_1(0) = 0$ and $\gamma_2(\pi) = 0$ then the slopes of $g(\Gamma_1)$ and $g(\Gamma_2)$ at $g(0)$ are both 0. As Γ_1 and Γ_2 are at right angles at 0 then g is not conformal at 0.

Question 10

(a) 10 marks

(a)(i) f has singularities at z = 0 and z = i. As $\lim_{z \to 0}(z-0)f(z) = 2i$ and $\lim_{z \to i}(z-i)f(z) = -2i$ then these are simple poles.

(a)(ii) $f(z) = \dfrac{2}{z(z-i)} = \dfrac{2i}{z(1+iz)}$

$$= \frac{2i}{z}\left\{\sum_{n=0}^{\infty}(-iz)^n\right\}$$

since $|iz| < 1$ on $\{z : 0 < |z| < 1\}$ (Unit B3, Section 3, Para. 5)

Hence the required Laurent series about 0 is

$$2\sum_{n=0}^{\infty}(-iz)^{n-1} = \frac{2i}{z} + 2 - 2iz + 2z^2 - \ldots + 2(-iz)^{n-1} + \ldots .$$

(a)(iii) $f(z) = \dfrac{2}{z(z-i)} = \dfrac{1}{\{(z-i)+i\}}\dfrac{2}{(z-i)} = \dfrac{2}{(z-i)^2}\dfrac{1}{1+\frac{i}{z-i}}$

$$= \frac{2}{(z-i)^2}\left\{\sum_{n=0}^{\infty}\left(\frac{-i}{z-i}\right)^n\right\}$$

since $|i / (z-i)| < 1$ on $\{z : |z - i| > 1\}$ (Unit B3, Section 3, Para. 5)

Therefore the required Laurent series about i is

$$-2\sum_{n=0}^{\infty}\left(\frac{-i}{z-i}\right)^{n+2} = \frac{2}{(z-i)^2} - \frac{2i}{(z-i)^3} - \frac{2}{(z-i)^4} - \ldots - 2\left(\frac{-i}{z-i}\right)^{n+2} - \ldots .$$

(b) 8 marks

(b)(i) The Laurent series for $g(z) = z^2 \sin(1/z)$ about 0 is

$$z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n-1}$$

Therefore the required series for g is $z - \frac{1}{6z} + \frac{1}{120z^3} - \ldots \qquad z \in \mathbb{C} - \{0\}$

(b)(ii) g has an essential singularity at 0 since there are an infinite number of terms with negative powers of z. (Unit B4, Section 2, Para. 8)

(b)(iii) $z^2 \sin(1/z)$ is analytic on the punctured disc $\mathbb{C}$ - {0}.

As C is a circle with centre 0 then (Unit B4, Section 4, Para. 2)

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz = 2\pi i\, a_{-1} = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}$$

where a_{-1} is the coefficient of z^{-1} in the Laurent series for g about 0.

(b)(iv)

$z^{2n} \sin(1/z)$ (n = 1, 2, 3, ..) is analytic on the punctured disc $\mathbb{C}$ - {0}.

The Laurent series about 0 for $z^{2n} \sin(1/z)$ on this disc is

$$z^{2n} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left(\frac{1}{z}\right)^{2m+1} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left(\frac{1}{z}\right)^{2(m-n)+1} = \sum_{s-\infty}^{\infty} a_s z^s$$

As C is a circle with centre 0 then (Unit B4, Section 4, Para. 2)

$$\int_C z^{2n} \sin\left(\frac{1}{z}\right) dz = 2\pi i\, a_{-1} = 2\pi i\left(\frac{(-1)^n}{(2n+1)!}\right), \quad \text{for } n = 1, 2, 3, \ldots$$

Question 11

(a) 6 marks

Since $f(z) = \dfrac{\pi \cot \pi z}{9(z - 2i/3)(z + 2i/3)}$ then f has simple poles at z = ±2i/3.

By the cover-up rule (Unit C1, Section 1, Para. 3)

$$\mathrm{Res}(f, 2i/3) = \frac{\pi \cot(2i\pi/3)}{9(2i/3 + 2i/3)} = \frac{\pi \cot(2i\pi/3)}{12i}, \text{ and}$$

$$\mathrm{Res}(f, -2i/3) = \frac{\pi \cot(-2i\pi/3)}{9(-2i/3 - 2i/3)} = \frac{\pi \cot(-2i\pi/3)}{-12i}.$$

Since sin(iz) = i sinh z and cos(iz) = cosh z then cot(iz) = - i coth(z).

Therefore $\mathrm{Res}(f, 2i/3) = -\dfrac{\pi \coth(2\pi/3)}{12}$ and

$$\mathrm{Res}(f, -2i/3) = \frac{\pi \coth(-2\pi/3)}{12} = -\frac{\pi \coth(2\pi/3)}{12}.$$ (Unit A2, Section 4, Para. 6)

f(z) = g(z) / h(z) where $g(z) = \dfrac{\pi \cos \pi z}{9z^2 + 4}$ and h(z) = sin πz.

g and h are analytic at 0, h(0) = 0, and $h'(0) = \pi \cos(0) = \pi \neq 0$.

Therefore by the g/h rule (Unit C1, Section 1, Para. 2)

$$\mathrm{Res}(f, 0) = \frac{g(0)}{h'(0)} = \frac{\pi * 1}{4} * \frac{1}{\pi} = \frac{1}{4}.$$

[You could also use Unit C1, Section 4, Para 1 – last line]

(b) 8 marks

The method given in Unit C1, Section 4, Para. 1 will be used.

$f(z) = \pi \cot \pi z * \phi(z)$ where $\phi(z) = 1/(9z^2 + 4)$.

ϕ is an even function which is analytic on $\mathbb{C}$ except for simple poles at the non-integral points $z = \pm 2i/3$.

Let S_N be the square contour with vertices at $(N + ½)(\pm 1 \pm i)$.

On S_N we have $|z| \geq N + ½$ so, using the backwards form of the Triangle Inequality (Unit A1, Section 5, Para. 2),

$$|9z^2 + 4| \geq |\, |9z^2| - 4 \,| \geq 9(N + ½)^2 - 4 \geq 9N^2.$$

On S_N we also have $\cot \pi z \leq 2$ (Unit C1, Section 4, Para. 2) so on C_N

$$|\, f(z) \,| \leq \frac{\pi(2)}{9N^2}.$$

The length of the contour S_N is $4(2N + 1)$.

As f is continuous on the contour S_N then by the Estimation Theorem (Unit B1, Section 4, Para. 3) we have

$$\left| \int_{S_N} f(z)dz \right| \leq \frac{2\pi}{9N^2} 4(2N+1) = \frac{8\pi}{9N}\left(2 + \tfrac{1}{N}\right).$$

Hence $\lim_{N \to \infty} \left| \int_{S_N} f(z)dz \right| = 0$.

Therefore the conditions specified in Unit C1, Section 4, Para. 1 hold so

$$\sum_{n=1}^{\infty} \frac{1}{9n^2 + 4} = -\frac{1}{2}\left(\operatorname{Res}(f,0) + \operatorname{Res}(f, 2i/3) + \operatorname{Res}(f, -2i/3)\right)$$
$$= -\frac{1}{8} + \frac{\pi}{12} \coth \frac{2\pi}{3}.$$

(c) 4 marks

$$\sum_{n=-\infty}^{\infty} \frac{1}{9n^2 + 4} = \sum_{n=-\infty}^{-1} \frac{1}{9n^2 + 4} + \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{9n^2 + 4}$$
$$= \frac{1}{4} + 2\sum_{n=1}^{\infty} \frac{1}{9n^2 + 4} = \frac{\pi}{6} \coth \frac{2\pi}{3}.$$

Question 12

(a) 8 marks

(a)(i) The circle C has centre $\lambda = 0$, and radius $r = 2$. I shall take $\alpha = 1 + i$ as an inverse point with respect to the circle C and show that the corresponding inverse point $\beta = 2(1 + i)$.

Since C is not an extended line then $k \neq 1$. Therefore the equation $(\alpha - \lambda)\overline{(\beta - \lambda)} = r^2$ given in Unit D1, Section 3, Para. 7 holds.

Hence $(1 + i)\overline{(\beta)} = 4$. Taking the conjugate of both sides gives $(1-i)\,\beta = 4$ or $\beta = 2(1 + i)$..

Hence the given α and β are inverse point with respect to C.

(a)(ii)

$$g(\alpha) = \frac{2}{(1+i)-(1+i)} = \infty\text{, and } g(\beta) = \frac{2}{2(1+i)-(1+i)} = 1 - i$$

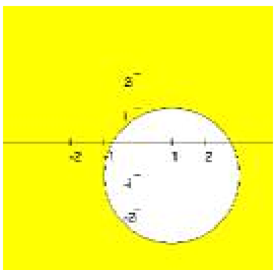
As α and β are inverse points with respect to the generalised circle C then $\hat{g}(\alpha)$ and $\hat{g}(\beta)$ are inverse points with respect to $\hat{g}(C)$. (Unit D1, Section 3, Para. 6)

Therefore the centre of the circle $\hat{g}(C)$ is at $1 - i$ (Unit D1, Section 3, Para. 5). Since a point on C is mapped to a point on $\hat{g}(C)$ then $g(2) = \frac{2}{2-(1+i)} = \frac{2}{1-i} = 1 + i$ is on $\hat{g}(C)$. Therefore the radius of $\hat{g}(C)$ is $|(1 + i) - (1 - i)| = 2$.

The image of C under G is the boundary of the white circle in the diagram below.

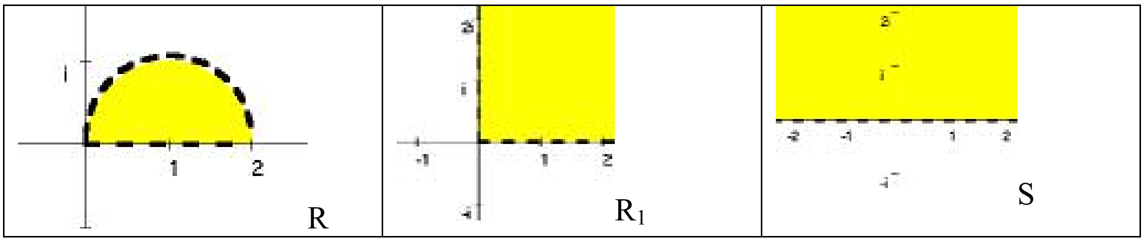
(a)(iii)

As $g(1 + i) = \infty$ then a point inside the circle C is mapped to a point outside $\hat{g}(C)$. Therefore D is the open shaded region with boundary $\hat{g}(C)$.



(b) 10 marks

(b)(i)



(b)(ii)

Using the formula for a transformation mapping points to the standard triple (Unit D1, Section 2, Para. 11) then the Möbius transformation $\hat{f}_1$ which maps 0, 1, and 2 to 0, 1, and ∞ respectively is

$$f_1(z) = \frac{(z-0)}{(z-2)}\frac{(1-2)}{(1-0)} = \frac{-z}{z-2}$$

Therefore the boundaries of R are mapped to extended lines in R_1. Since Möbius transformations are conformal these lines in R_1 meet at the origin at right-angles.

The line along the origin in R is mapped to the positive real-axis in R_1 since $f(1) = 1$. As we move from 0 to 1 in R the region to be mapped is on the left-hand side. As the transformation is conformal this must also be the case in R_1. Therefore is mapped to R_1 by f_1.

(b)(iii) $w = z_1^2$ is a conformal mapping from R_1 to S.

Therefore a conformal mapping from R to S is

$$f(z) = \left(\frac{-z}{z-2}\right)^2.$$

(b)(iv) The point 0 belongs to the closure of R. Since at this point $f^{/}(z) = 0$ then f is not conformal (Unit A4, Section 4, Para. 6).