## *2003 Question 1*

**(a) 2 marks**

$(1 - i)^2 = 1 - 2i + i^2 = -2i.$
$(1 - i)^4 = -4.$

Therefore $\dfrac{1}{(1-i)^4} = -\dfrac{1}{4}.$

Alternatively $1 - i = \sqrt{2}\exp(-i\pi/4)$, $(1-i)^4 = 4\exp(-i\pi) = -4$ .

**(b) 2 marks**

$i = \exp(i\pi/2).$

Square roots of $i$ are $\pm\exp(i\pi/4) = \pm\dfrac{1}{\sqrt{2}}(1+i).$

**(c) 2 marks**

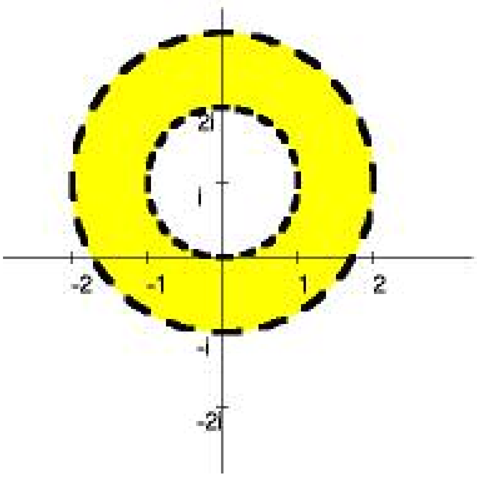
$\text{Log}(-2) = \log_e|-2| + i\,\text{Arg}(-2) = \log_e 2 + i\pi.$

**(d) 2 marks**

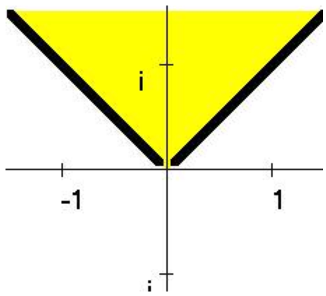
$(-2)^i = \exp(i\,\text{Log}(-2)) = \exp(i\log_e 2 - \pi) = \exp(-\pi)\{\cos(\log_e 2) + i\sin(\log_e 2)\}$

## *2003 Question 2*

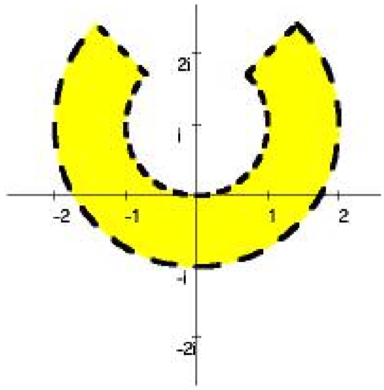
**(a) 3 marks**




A




B




C = A - B

Note origin not included in B as Arg not defined there.

**(b) 4 marks**

**(b)(i)** A and C.
**(b)(ii)** C.
**(b)(iii)** B

**(c) 1 mark**

$\{ z : \mathrm{Re}(z) \geq 0\}$ or $\{z : |z| \geq 1\}$.

## *2003 Question 3*

(a)   3 marks

(a)(i) The standard parametrization (Unit A2, Section 2, Para. 3) for $\Gamma$ is

$$\gamma(t) = (1-t) + it, \qquad t \in [0, 1].$$

(a)(ii)

As $\gamma$ is differentiable on [0, 1], $\gamma'$ is continuous on [0, 1], and $\gamma'$ is non-zero on [0, 1] then $\gamma$ is a smooth path (Unit A4, Section 4, Para. 3).

As $\gamma$ is a smooth path then (Unit B1, Section 2, Para. 1)

$$\begin{aligned}\int_\Gamma \operatorname{Im} z\, dz &= \int_0^1 \operatorname{Im}(\gamma(t))\,\gamma'(t)\,dt \\ &= \int_0^1 t\,(-1+i)\,dt \\ &= (-1+i)\left[\frac{t^2}{2}\right]_0^1 = \frac{-1+i}{2}.\end{aligned}$$

(b)   5 marks

The length of the contour $\Gamma$, $L = |1 - i| = \sqrt{2}$.

Using the Triangle Inequality (Unit A2, Section 5, Para. 2a) then for z on the contour $\Gamma$

$$\begin{aligned}|\cosh z| &= \tfrac{1}{2}\left|e^z + e^{-z}\right| \quad \text{(Unit A2, Section 4, Para. 6)} \\ &\le \tfrac{1}{2}\left(\left|e^z\right| + \left|e^{-z}\right|\right) = \tfrac{1}{2}\left(e^{\operatorname{Re} z} + e^{-\operatorname{Re} z}\right) \quad \text{(Unit A2, Section 4, Para. 2b)} \\ &< \tfrac{1}{2}\left(e^1 + e^1\right) = e \quad .\end{aligned}$$

Using the Backwards form of the Triangle Inequality (Unit A2, Section 5, Para. 2b) then for $z \in \Gamma$

$$\left|4 + z^2\right| \ge \left|4 - \left|-z^2\right|\right| \ge |4 - 1| = 3.$$

Putting $f(z) = \dfrac{\cosh z}{4 + z^2}$ then on $\Gamma$ we have $|f(z)| \le \dfrac{e}{3} = M$.

By the Quotient Rule (Unit A3, Section 2, Para. 5) f(z) is continuous on $\{z \in \mathbb{C} : |z| > 1\}$ and hence on the contour C. Therefore by the Estimation Theorem (Unit B1, Section 4, Para. 3)

$$\left|\int_C \frac{\cosh z}{4 + z^2}\,dz\right| \le ML = \frac{e}{3}\sqrt{2}.$$

## *2003 Question 4*

(a) 3 marks

$\mathbb{C}$ is a simply-connected region and $1/z^3$ is analytic on $\mathbb{C}$ except for a singularity at 0. C is simple-closed contour in $\mathbb{C}$ which does not pass through the singularity. Therefore by Cauchy's Residue Theorem (Unit C1, Section 2, Para. 1) we have

$$\int_C \frac{1}{z^3}\,dz = \operatorname{Res}\left(\frac{1}{z^3}, 0\right) = 0.$$

(b) 3 marks

Let g(z) = cos(z- π). g is a function which is analytic on the simply-connected region $\mathbb{C}$ (Unit B2, Section 1, Para. 3).

The contour C is a simple-closed contour in $\mathbb{C}$. Since $z^3$ is zero inside the circle C then using Cauchy's $n^{th}$ Derivative Formula (Unit B2, Section 3, Para. 1), with n = 2 and α = 0 we have

$$\int_C \frac{\cos(z-\pi)}{z^3}\,dz = \int_C \frac{g(z)}{z^3}\,dz = \frac{2\pi i}{2!} g^{(2)}(0)$$

$g'(z) = -\sin(z-\pi)$.

$g''(z) = -\cos(z-\pi)$.

So $g''(0) = -\cos(-\pi) = 1$.

Hence $\int_C \frac{\cos(z-\pi)}{z^3}\,dz = \pi i$.

(b) 2 marks

R = {Z : |z| < π} is a simply-connected region and $\frac{\cos z}{(z-\pi)^3}$ is analytic on R. Since the contour C lies in R then by Cauchy's Theorem (Unit B2, Section 1, Para. 4)

$$\int_c \frac{\cos z}{(z-\pi)^3}\,dz = 0.$$

## *2003 Question 5*

(a) 3 marks

$$f(z) = \frac{z^2+1}{2z(z+\frac{1}{2})(z+2)}$$

f is an analytic function with simple poles at z = 0, -½, and -2. Therefore (Unit C1, Section 1, Para. 1).

$$\mathrm{Res}(f,0) = \lim_{z\to 0}(z-0)f(z) = \frac{1}{2(\frac{1}{2})(2)} = \frac{1}{2},$$

$$\mathrm{Res}(f,-\tfrac{1}{2}) = \lim_{z\to -\frac{1}{2}}(z+\tfrac{1}{2})f(z) = \frac{(-\frac{1}{2})^2+1}{2(-\frac{1}{2})(-\frac{1}{2}+2)} = \frac{\frac{5}{4}}{-\frac{3}{2}} = -\frac{5}{6}, \text{ and}$$

$$\mathrm{Res}(f,0) = \lim_{z\to -2}(z+2)f(z) = \frac{2^2+1}{2(-2)(-2+\frac{1}{2})} = \frac{5}{6}.$$

(b) 5 marks

I shall use the strategy given in Unit C1, Section 2, Para. 2.

$$\int_0^{2\pi}\frac{\cos t}{5+4\cos t}\,dt = \int_C \frac{\frac{1}{2}(z+z^{-1})}{5+4(\frac{1}{2})(z+z^{-1})}\frac{1}{iz}\,dz \qquad \text{, where C is the unit circle } \{z : |z| = 1\}.$$

$$= -\frac{i}{2}\int_C \frac{z^2+1}{z(5z+2z^2+2)}\,dz$$

$$= -\frac{i}{2}\int_C \frac{z^2+1}{z(2z+1)(z+2)}\,dz$$

f is analytic on the simply-connected region $\mathbb{C}$ except for a finite number of singularities. C is a simple contour in $\mathbb{C}$ not passing through any of the singularities. Since the singularities at z = -½, and 0 are inside the circle C then by Cauchy's Residue Theorem (Unit C1, Section 2, Para. 1) we have

$$\int_0^{2\pi}\frac{\cos t}{5-4\cos t}\,dt = -\frac{i}{2}*2\pi i\{\mathrm{Res}\ (f,0)+\mathrm{Res}\ (f,-\tfrac{1}{2})\}$$

$$= \pi\left\{\frac{1}{2}-\frac{5}{6}\right\} = -\frac{\pi}{3}.$$

## *2003 Question 6*

(a) 4 marks

(a)(i) $\Gamma(5) = 4! = 24$. (Unit C3, Section 4, Para. 2b).

(a)(ii) $\Gamma(5/2) = 3/2\ \Gamma(3/2)$ (Unit C3, Section 4, Para. 2c).

$= \frac{3}{4}\sqrt{\pi}$. (Unit C3, Section 4, Para. 4).

(a)(iii)

Since the functional equation of the Gamma function (Unit C3, Section 4, Para. 2) holds on $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$ (Unit C3, Section 4, Para. 3) so

$$\Gamma(i + 1) = i\,\Gamma(i) = i\,(i - 1)\,\Gamma(i - 1).$$

So $\frac{\Gamma(i+1)}{\Gamma(i-1)} = i(i-1) = -1 - i$.

(b) 4 marks

I shall use Weierstrass' M-test (Unit C3, Section 3, Para. 5) with $\phi_n(z) = \frac{e^z}{n^2}$ where n is an integer.

On E, $|\phi_n(z)| = \left|\frac{e^z}{n^2}\right| \le \frac{e^{|\mathrm{Re}\,z|}}{n^2}$ (Unit A2, Section 4, Para. 2b)

$\le \frac{e}{n^2}$ as $|\mathrm{Re}\,z| \le 1$ on E.

Therefore the 1st assumption of Weierstrass' M test holds if we set $M_n = \frac{e}{n^2}$.

Since $\sum_{n=1}^{\infty} M_n = e\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (Unit B3, Section 1, Para. 8) then $\sum_{n=1}^{\infty} M_n$ is convergent. Therefore the 2nd assumption of the M test also holds.

Hence by the M-test $\sum_{n=1}^{\infty}\frac{e^z}{n^2}$ converges uniformly on E.

## *2003 Question 7*

(a) 1 mark

q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C} - \{0\}$ and the conjugate velocity $\overline{q}(z) = 1/z$ is analytic on $\mathbb{C} - \{0\}$. Therefore (Unit D2 Section 1, Para. 14) q is a model fluid flow on $\mathbb{C} - \{0\}$.

(b) 5 marks

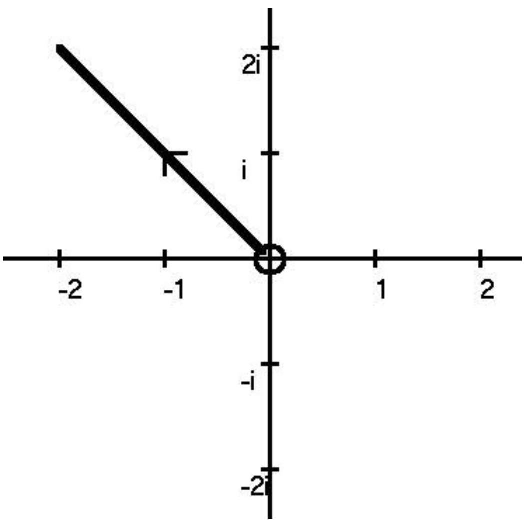
The complex potential function $\Omega$ is a primitive of $\overline{q}(z)$ (Unit D2, Section 2, Para. 1). Therefore the complex potential function $\Omega(z) = \text{Log}\, z$ and the stream function

$\Psi(x, y) = \text{Im}\,\Omega(z)$ (Unit D2, Section 4, Para. 4)

$= \text{Arg}\, z$ (Unit A2, Section 5, Para. 1)

A streamline through i is given by $\text{Arg}\, z = \Psi(-1,1) = 3\pi/4$.

The velocity function at i is q(-1+i) = $\dfrac{1}{-1-i} = \dfrac{-1+i}{2}$. (To the north-west).




(c) 2 marks

Flux of q across the unit circle $\Gamma = \{z : |z| = 1\}$ is (Unit D2, Section 1, Para. 10)

$$\text{Im}\left(\int_\Gamma \overline{q}(z)dz\right) = \text{Im}\left(\int_\Gamma \frac{1}{z}\,dz\right) = \text{Im}(2\pi i) = 2\pi$$

by Cauchy's Integral Formula (Unit B2, Section 2, Para 1).

[[ Alternatively use Unit D2, Section 2, Para. 1. ]]

## *2003 Question 8*

(a) 2 marks

Using the result in Unit D3, Section 2, Para. 1 then the iteration sequence $z_{n+1} = z_n^2 + 4z_n + 3$ is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + (1*3 + 4 / 2 - 4^2 / 4) = w_n^2 + 1$$

and conjugating function $h(z) = z + 2$.

Therefore $w_0 = h(z_0) = z_0 + 2 = -2 + 2 = 0$. (Unit D3, Section 1, Para. 7).

(b) 3 marks

If $\alpha$ is a fixed point of $P_1$ (Unit D3, Section 1, Para. 3) then
$P_1(\alpha) = \alpha^2 + 1 = \alpha$.

The solutions of $\alpha^2 - \alpha + 1 = 0$ are $\dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1 \pm i\sqrt{3}}{2}$.

$P_1{}'(z) = 2z$..

When $z = \dfrac{1 \pm i\sqrt{3}}{2}$ then $|P_1{}'(z)| = |1 \pm i\sqrt{3}| = 2$. Therefore $\dfrac{1 \pm i\sqrt{3}}{2}$ are repelling fixed points (Unit D3 Section 1, Para. 5).

(c) 3 marks

$P_1(0) = 1$, $P_1^2(0) = 1^2 + 1 = 2$, $P_1^3(0) = 2^2 + 1 = 5$.

Since $|P_1^3(0)| > 2$ then 1 does not belong to the Mandelbrot set (Unit D3, Section 4, Para. 5). Therefore the keep set $K_1$ is not connected (Unit D3, Section 4, Para. 3). Hence by the Fatou-Julia Theorem $0 \notin K_1$.

## *2003 Question 9*

(a) 7 marks

(a)(i)

The derivative of f (Unit A4, Section 1, Para. 1) at 0 is

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z \to 0} \frac{\overline{z}}{z} \quad \text{if this limit exists.}$$

If $z \to 0$ along the real-axis then $\lim_{z \to 0} \frac{\overline{z}}{z} = \lim_{z \to 0} \frac{z}{z} = 1$.

If $z \to 0$ along the imaginary-axis then $\lim_{z \to 0} \frac{\overline{z}}{z} = \lim_{z \to 0} \frac{-z}{z} = -1$.

Therefore the limit does not exist and so f is not differentiable at 0.

(a)(ii)

Putting z = x + iy we have
f(x + iy) = x – iy = u(x, y) + i v(x, y)
where u(x,y) = x, and v(x,y) = - y.

$$\frac{\partial u}{\partial x}(x,y) = 1, \quad \frac{\partial u}{\partial y}(x,y) = 0,$$
$$\frac{\partial v}{\partial x}(x,y) = 0, \quad \frac{\partial v}{\partial y}(x,y) = -1.$$

If f is differentiable the Cauchy-Riemann equations (Unit A4, Section 2, Para. 1) hold.

As $\frac{\partial u}{\partial x}(0,0) = 1 \neq \frac{\partial v}{\partial y}(0,0) = -1$ then the Cauchy-Riemann equations do not hold at z = 0, and so f is not differentiable at 0.

(b) 11 marks

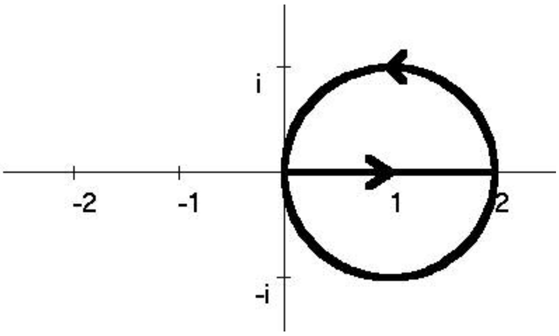
(b)(i) $g'(z) = 6iz$ on $\mathbb{C}$.

Since g is analytic on $\mathbb{C}$ and $g'(z) \neq 0$ on $\mathbb{C} - \{0\}$, then g is also conformal on $\mathbb{C} - \{0\}$ (Unit A4, Section 4, Para. 6).

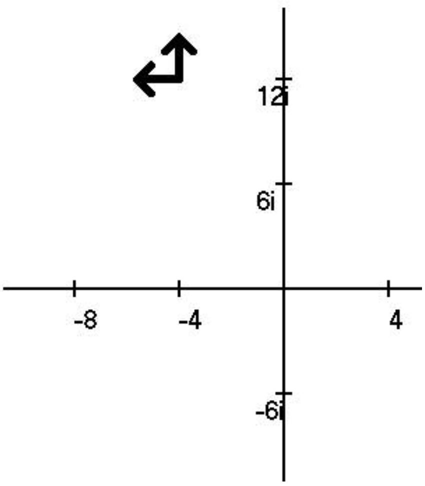
(b)(ii) $g(2) = -4 + 12i$ and $g^{/}(2) = 12i$.

A small disc centred at 2 with be mapped to a disc centred at -4 + 12i, rotated by an angle of Arg $g^{/}(2) = \pi/2$, and scaled by a factor of $| g^{/}(z) | = 12$.

(b)(iii)




(b)(iv)




$g(\Gamma_1)$ is the upward arrow.

(b)(v)

$g(\gamma_1(t)) = 12it^2 -4$ where $t \in [0, 1]$. $g^{/}(t) = 24it$.
As $\gamma_1(t) = 0$ when $t = 0$ then the slope of $g(\Gamma_1)$ at t is 0.

$g(\gamma_2(t)) = 3i(1 + e^{it})^2 -4$ where $t \in [0, \pi]$. $g^{/}(t) = -6e^{it}(1 + e^{it})$.
As $\gamma_2(t) = 0$ when $t = \pi$ then the slope of $g(\Gamma_2)$ at t is 0.

From the diagram in part (b)(iii) it can be seen that the paths $\Gamma_1$ and $\Gamma_2$ are at right-angles at 0. Since the directions of the $g(\Gamma_1)$ and $g(\Gamma_2)$ are not also at right-angles there then g is not conformal at 0.

### *2003 Question 10*

(a) 5 marks

The singularities occur when the denominator of f is zero. Therefore there are singularities at z = 0 and z = i.

As $\lim_{z \to 0} zf(z) = \frac{\sin 0}{(-i)^3} = 0$ then the singularity at 0 is removable (Unit B4, Section 3, Para. 1(D)).

As $\lim_{z \to i} (z-i)^3 f(z) = \frac{\sin i}{i} \neq 0$ then f has a pole of order 3 at i. (Unit B4, Section 3, Para. 2(B)).

(b) 6 marks

(b)(i)

$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$ , for $z \in \mathbf{C}$. (Unit B3, Section 3, Para. 5)

Therefore the Laurent series for g(z) = exp(-1/z) is

$$1 - \frac{1}{z} + \frac{1}{2!z^2} - \frac{1}{3!z^3} + \ldots + \frac{(-1)^n}{n!z^n} + \ldots = \sum_{n=0}^{\infty} z_n$$

As $\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}}{(n+1)!z^{n+1}} \frac{n!z^n}{(-1)^n}\right| = \frac{1}{(n+1)|z|}$ tends to 0 as $n \to \infty$ then by the Ratio Test (Unit B3, Section 1, Para. 15) then this series converges when $z \neq 0$.

Therefore the annulus of convergence is $\mathbb{C}$ - {0}.

(b)(ii)

Using the series for g found above then the coefficient of $z^{-1}$ in $z^4$ exp(-1/z) is -1/5!.

As C is a circle with centre 0 then (Unit B4, Section 4, Para. 2)

$$\int_C z^4 \exp(-1/z)dz = 2\pi i\left(-\frac{1}{5!}\right) = -\frac{\pi i}{40} \ .$$

(b) 5 marks

The Taylor series around 0 for cosh z and Log(1 + z) are

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots \qquad \text{for } z \in \mathbb{C},$$

and $$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots \qquad \text{for } |z| < 1.$$

h(z) = Log (cosh z) = Log (1 + [cosh z - 1]).

At z = 0 then (cosh z - 1) = 0. Therefore we can expand the series about 0 using the Composition Rule (Unit B3, Section 4, Para. 3) when |cosh z - 1| < 1.

$$\begin{aligned}
h(z) &= \left(\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots\right) - \frac{1}{2}\left(\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots\right)^2 + \frac{1}{3}\left(\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots\right)^3 + \ldots \\
&= \frac{z^2}{2} + z^4\left(\frac{1}{4!} - \frac{1}{2}\frac{1}{(2!)^2}\right) + z^6\left(\frac{1}{6!} + \frac{1}{2!\,4!} + \frac{1}{3(2!)^2}\right) + \ldots \\
&= \frac{z^2}{2} + z^4\left(\frac{1}{24} - \frac{1}{8}\right) + z^6\left(\frac{1}{720} - \frac{1}{48} + \frac{1}{24}\right) + \ldots \\
&= \frac{z^2}{2} - \frac{z^4}{12} + \frac{z^6}{720}(1 - 15 + 30) + \ldots \\
&= \frac{z^2}{2} - \frac{z^4}{12} + \frac{z^6}{45} + \ldots
\end{aligned}$$

$$h'(z) = \frac{\sinh z}{\cosh z} = \tanh z.$$

Therefore using the Differentiation Rule (Unit B3, Section 2, Para. 9) we have

$$\tanh z = z - \frac{z^3}{3} + \frac{2z^5}{15} + \ldots$$

## *2003 Question 11*

(a) 10 marks

(a)(i)

Let $g(z) = 101$.

Using the Triangle Inequality (Unit A1, Section 5, Para. 2) then on the simple-closed contour $\{ z : |z| = 2\}$ we have

$$| f(z) - g(z) | = | z^6 + 9z^2 | \leq | z^6 | + 9 | z^2 | = 64 + 36 = 100$$
$$< 101 = | g(z) |$$

As f and g are analytic (Unit A4, Section 1, Para. 7) on the simply-connected region $\mathbb{C}$, and $\Gamma$ is a simple-closed contour in $\mathbb{C}$ then by Rouché's Theorem (Unit C2, Section 2, Para. 4) f has the same number of zeros inside $\Gamma$ as g.
Therefore $f(z) = 0$ has no solutions in the disc $\{ z : |z| < 2\}$.

On $|z| = 2$ then

$$| z^6 + 9z^2 + 101 | \geq | 101 - | z^6 + 9z^2| | \geq 101 - 100 = 1$$

then there are no zeros on $|z| = 2$.

Therefore $f(z) = 0$ has no solutions in $\{ z : |z| \leq 2\}$.

.

(a)(ii) If z is real then $z^6 \geq 0$ and $z^2 \geq 0$. As all the terms in f(z) are non-negative then $f(z) \geq 101$. Therefore there are no zeros on the real axis.

(a)(iii) If $f(z) = 0$ then $\overline{f(z)} = \overline{z^6 + 9z^2 + 101} = (\bar{z})^6 + 9(\bar{z})^2 + 101 = 0$. Therefore if z is a root of f then so is it's complex conjugate $\bar{z}$. Since none of the 6 roots are real, by part (a)(ii), then this means that 3 of them lie above the real axis (Im z > 0) and 3 below. Since there are no solutions in $\{ z : |z| \leq 2\}$ then f has exactly 3 zeros in $\{z : |z| > 2, \ \text{Im}\, z > 0\}$.

Since $f(-z) = f(z)$ then if z is a root of $f(z) = 0$ then $-z$ is also a root together with their conjugates $\bar{z}$ and $-\bar{z}$. If z is not real or purely imaginary then these 4 values are distinct, and one of them lies in each of the 4 regions bounded by the real and imaginary axes. Since we know that 2 of the roots are on the imaginary axis then exactly one of the other roots occurs in each of the 4 specified regions.

(b) 8 marks

I shall use the result in Unit C1, Section 3, Para. 9.

Let p(t) = 1, q(t) = t (1 + $t^2$) = t (t - i)(t + i), and $r(t) = \frac{p(t)}{q(t)}\exp(ikt)$, where k = 1.

Since (1) the degree of q exceeds that of p by more than 1.
(2) the only pole of p/q on the real axis (at 0) is simple,

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\exp(ikt)\,dt = 2\pi i S + \pi i T$$

where S is the sum of the residues of the function r(z) at those poles in the upper half-plane, and T is the sum of the residues of the function r(z) at those poles on the real axis.

The only pole in the upper half-plane is at z = i and

$$S = \text{Res}\,(r, i) = \lim_{z \to i} (z-i)r(z)$$

$$= \frac{\exp(i^2)}{i(i+i)} = \frac{e^{-1}}{-2} = -\frac{1}{2e}.$$

$$T = \text{Res}\,(r, 0) = \lim_{z \to 0} (z-0)r(z) = \frac{1}{1}e^0 = 1.$$

Hence $\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\exp(ikt)\,dt = 2\pi i\left\{-\frac{1}{2e}\right\} + \pi i(1).$

Therefore taking the imaginary part gives

$$\int_{-\infty}^{\infty} \frac{\sin t}{t(1+t^2)}\,dt = \pi - \frac{\pi}{e}.$$

### *2003 Question 12*

(a) 6 marks

(a)(i) $\beta = -1$. (Unit D1, Section 3, Para. 7b)

$g(1) = 0$ and $g(-1) = \infty$.

(a)(ii)

Inverse points are mapped to the inverse points. (Unit D1, Section 3, Para. 6). As 0 and $\infty$ are inverse points then 0 is the centre of a circle (Unit D1, Section 3, Para. 5).

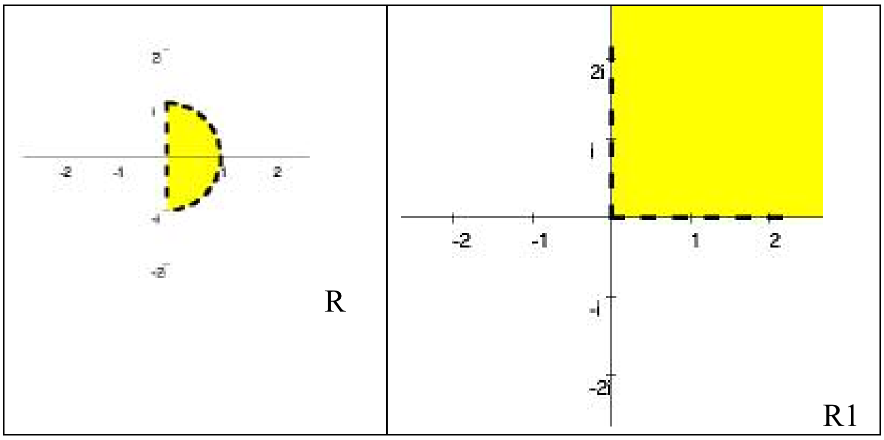
As boundary points are mapped to boundary points then $g(0) = -1$ is on the boundary of the circle. (Unit D1, Section 4 Para. 3). As -1 is on the circle it has radius 1. Therefore the image of the real axis is the unit circle.

(a)(iii)

As 1 is in the specified region $\{z : \text{Re } z > 0\}$ then $g(1) = 0$ is in the image of the region. Therefore the image of the region is the interior of the unit circle.

(b ) 10 marks

(b)(i)



(b)(ii)

Map +i to 0, 0 to 1, and -i to $\infty$.

Using the result for mapping to the standard triple (Unit D1, Section 2, Para. 11) a suitable transformation is

$$f_1(z) = \frac{(z-i)(0+i)}{(z+i)(0-i)} = -\frac{z-i}{z+i}.$$

At z = i the angle between the boundary lines of R are at an angle of $\pi/2$. Therefore as the transformation is conformal then this is also the angle at the origin of the transformed lines. Going along the straight line boundary in R from i towards 0 the region to be mapped is on the left. Therefore the image of the region is above the non-negative real axis. Therefore f maps R to $R_1$.

(b)(iii)

$h(z) = -iz^2$ maps $R_1$ to the region $\{z : \text{Re } z > 0\}$.

Therefore a conformal mapping from R to D is

$$(g \circ h \circ f_1)(z) = (g \circ h)\left(-\frac{z-i}{z+i}\right) = g\left(-i\left(-\frac{z-i}{z+1}\right)^2\right) = \frac{-i\left(-\frac{z-i}{z+1}\right)^2 - 1}{-i\left(-\frac{z-i}{z+1}\right)^2 + 1}.$$

(a) 2 marks

(c)(i) f(z) = z.

(c)(ii) $f(z) = e^z$.