2002 Question 1

(a) 2 marks

$\exp(2 + \pi i/6)$

$= e^2 \{\cos(\pi/6) + i\sin(\pi/6)\}$ (Unit A2, Section 4, Para. 1)

$= \frac{e^2}{2}\left(\sqrt{3} + i\right)$

(b) 2 marks

$\text{Log}(2 - 2i) = \log_e|2 - 2i| + i\,\text{Arg}(2 - 2i)$ (Unit A2, Section 5, Para. 1)

$= \log_e\left(2\sqrt{2}\right) - i\pi/4 = \tfrac{3}{2}\log_e 2 - i\pi/4$.

(c) 2 marks

$-i = \exp(-i\pi/2)$

Therefore the square roots of $-i$ are (Unit A1, Section 3, Para. 5) are

$\exp(-i\pi/4) = \cos(-\pi/4) + i\sin(-\pi/4) = \frac{1}{\sqrt{2}}(1 - i)$

and $-\exp(-i\pi/4) = \frac{1}{\sqrt{2}}(-1 + i)$

(d) 2 marks (Unit A2, Example 5.3(c))

$i^i = \exp(i\,\text{Log}\,i)$ (Unit A2, Section 5, Para. 3)

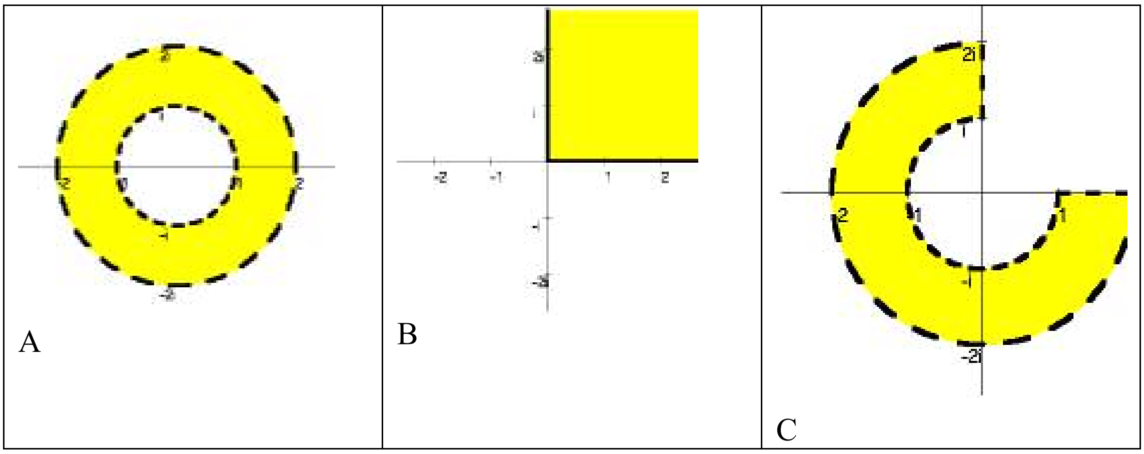
$= \exp(i\{\log_e|i| + i\,\text{Arg}\,i\})$ (Unit A2, Section 5, Para. 1)

$= \exp(i\{0 + i\pi/2\})$

$= \exp(-\pi/2)$

2002 Question 2

(a) 3 marks



[{0} is included in the definition of B as Arg z is not defined when z = 0 (Unit A1, Section 2, Para. 5).]

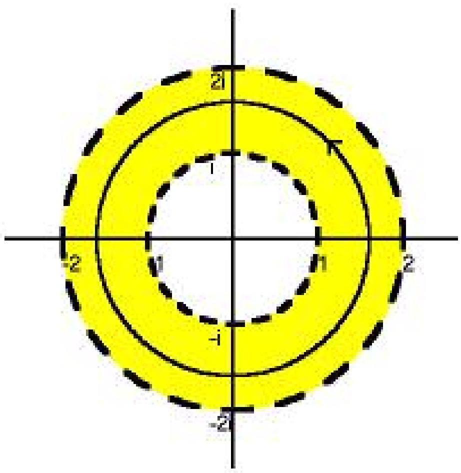
(b) 2 marks

A and C are regions (Unit A3, Section 4, Paras. 6 and 7).
B is not a region as it is not open.

(a) 2 marks

A is not simply-connected.
C is simply-connected. (Unit B2, Section 1, Para. 3).

(d) 1 mark



2002 Question 3

(a) 3 marks

The standard parametrization (Unit A2, Section 2, Para. 3) for C is

$\gamma(t) = 2(\cos t + i \sin t), \quad t \in [0, 2\pi] \quad (= 2e^{it}$ is easier. See 2000 Qu. 3)

and $\gamma'(t) = 2(-\sin t + i \cos t)$.

As γ is differentiable on $[0, 2\pi]$, γ' is continuous on $[0, 2\pi]$, and γ' is non-zero on $[0, 2\pi]$ then γ is a smooth path (Unit A4, Section 4, Para. 3).

As γ is a smooth path then (Unit B1, Section 2, Para. 1)

$$\begin{aligned}\int_C \bar{z}\, dz &= \int_0^{2\pi} \overline{2(\cos t + i \sin t)}\; 2(-\sin t + i \cos t)\, dt \\ &= 4\int_0^{2\pi} (\cos t - i \sin t)(-\sin t + i \cos t)\, dt \\ &= 4\int_0^{2\pi} i(\cos^2 t + \sin^2 t)\, dt = 4i\int_0^{2\pi} dt = 8\pi i\end{aligned}$$

(b) 5 marks

The length of the contour C, $L = 2\pi * 2 = 4\pi$.

Using the Triangle Inequality (Unit A2, Section 5, Para. 2a) then for z on the contour C

$$|\cos z| = \tfrac{1}{2}\left|e^{iz} + e^{-iz}\right| \quad \text{(Unit A2, Section 4, Para. 4)}$$

$$\le \tfrac{1}{2}\left(\left|e^{iz}\right| + \left|e^{-iz}\right|\right) = \tfrac{1}{2}\left(e^{-\operatorname{Im} z} + e^{\operatorname{Im} z}\right) \quad \text{(Unit A2, Section 4, Para. 2b)}$$

$$< \tfrac{1}{2}\left(e^2 + e^2\right) = e^2 \quad \text{as } |z| = 2.$$

Using the Backwards form of the Triangle Inequality (Unit A2, Section 5, Para. 2b) then for $z \in C$

$$\left|1 - z^3\right| \ge \left|1 - |z|^3\right| \ge |1 - 8| = 7$$

Putting $f(z) = \dfrac{\cos z}{1 - z^3}$ then on C we have $|f(z)| \le \dfrac{e^2}{7} = M$.

By the Quotient Rule (Unit A3, Section 2, Para. 5) f(z) is continuous on $\{z \in \mathbb{C} : |z| > 1\}$ and hence on the contour C. Therefore by the Estimation Theorem (Unit B1, Section 4, Para. 3)

$$\left|\int_C \frac{\cos z}{1 - z^3}\, dz\right| \le ML = \frac{e^2}{7} * 4\pi = \frac{4}{7}\pi e^2.$$

2002 Question 4

(a) 3 marks

f(z) has a pole of order 3 at z = -i.

Let ***R*** = {z : |z| < 1}. As ***R*** is a simply-connected region (Unit B2, Section 1, Para. 3) and f is analytic on ***R*** and C is a closed contour in ***R*** then by Cauchy's Theorem (Unit B2, Section 1, Para. 4)

$$\int_C f(z)dz = 0$$

(b) 5 marks

Let ***R*** = $\mathbb{C}$ and g(z) = z^3e^z. ***R*** is a simply-connected region (Unit B2, Section 1, Para. 3), g is analytic on ***R*** and C is a simple-closed contour in ***R***.

As –i lies inside C then by Cauchy's nth Derivative formula (Unit B2, Section 3, Para. 1) with n = 2 and α = -i, we have

$$\int_{C_2} \frac{z^3e^z}{(z+i)^3}dz = \frac{2\pi i}{2!}g^{(2)}(-i) = \pi i\, g^{(2)}(-i)$$

$g^{(1)}(z) = (3z^2 + z^3)\, e^z$
$g^{(2)}(z) = (\{6z + 3z^2\} + \{3z^2 + z^3\})\, e^z = (6z + 6z^2 + z^3)\, e^z$

$g^{(2)}(-i) = (-6i - 6 + i)\, e^{-i} = -(6 + 5i)\, e^{-i}$
$= -(6 + 5i)(\cos 1 - i \sin 1)$
$= -\{(6 \cos 1 + 5 \sin 1) + i\,(5 \cos 1 - 6 \sin 1)\}$

Therefore $\displaystyle\int_{C_2} \frac{z^3e^z}{(z+i)^3}dz = \pi\{(5\cos 1 - 6\sin 1) - i(6\cos 1 + 5\sin 1)\}$

[Seems a strange answer.]

2002 Question 5

(a) 3 marks

f is an analytic function with simple poles at z= -1/3, and z = -3.

$$\mathrm{Res}\left(f,-\tfrac{1}{3}\right)=\lim_{z\to -1/3}\left(z+\tfrac{1}{3}\right)f(z)=\frac{1}{3\left(-\tfrac{1}{3}+3\right)}=\frac{1}{8}.$$

Unit C1, Section 1, Para. 1

$$\mathrm{Res}(f,-3)=\lim_{z\to -3}(z+3)\,f(z)=\frac{1}{3(-3)+1}=-\frac{1}{8}.$$

(b) 5 marks

I shall use the strategy given in Unit C1, Section 2, Para. 2.

$$\int_0^{2\pi}\frac{1}{5+3\cos t}\,dt=\int_C\frac{1}{5+3\left(\tfrac{1}{2}\right)\left(z+z^{-1}\right)}\frac{1}{iz}\,dz$$

, where C is the unit circle $\{z : |z| = 1\}$.

$$=-2i\int_C\frac{1}{10z+3z^2+3}\,dz=-2i\int_C\frac{1}{(3z+1)(z+3)}\,dz$$

f is analytic on the simply-connected region $\mathbb{C}$ except for a finite number of singularities. C is a simple contour in $\mathbb{C}$ not passing through any of the singularities. Since only the singularity at z = -1/3 is inside the circle C then, by Cauchy's Residue Theorem (Unit C1, Section 2, Para. 1), we have

$$\int_0^{2\pi}\frac{1}{5+3\cos t}\,dt=-2i*2\pi i\,\mathrm{Res}\left(f,-\tfrac{1}{3}\right)=4\pi*\tfrac{1}{8}=\frac{\pi}{2}$$

2002 Question 6

(a) 3 marks

(a)(i)

A point in the half-plane *T* can be represented as in both $\mathbb{C}_\pi$ and $\mathbb{C}_{2\pi}$ (Unit C2, Section 1, Para. 5) as

$$z = r\,e^{i\theta} \qquad \text{where } 0 < \theta < \pi.$$

Therefore for $z \in T$ we have (Unit A2, Section 5, Para. 6)

$$\mathrm{Log}_\pi(z) = \log_e|z| + i\mathrm{Arg}_\pi(z) = \log_e|z| + i\mathrm{Arg}_{2\pi}(z) = \mathrm{Log}_{2\pi}(z)$$

(a)(ii)

$\mathrm{Log}_\pi(z)$ and $\mathrm{Log}_{2\pi}(z)$ have the domains $\mathbb{C}_\pi$ and $\mathbb{C}_{2\pi}$ respectively. Since $T \subset \mathbb{C}_\pi \cap \mathbb{C}_{2\pi}$ and

$\mathrm{Log}_\pi(z) = \mathrm{Log}_{2\pi}(z)$ when $z \in T$ (part (a)(i)

then $\mathrm{Log}_{2\pi}$ and Log_π are direct analytic continuations of each other. (Unit C3, Section 1, Para. 1)

(b) 5 marks (or see Unit C3, Problem 2.2a)

Let $V = \{z : \mathrm{Im}\, z < 0\}$.

A point in the half-plane *V* can be represented as in both $C_{2\pi}$ and $C_{3\pi}$ (Unit C2, Section 1, Para. 5) as

$$z = r\,e^{i\theta} \qquad \text{where } \pi < \theta < 2\pi.$$

Therefore for $z \in V$ we have (Unit A2, Section 5, Para. 6)

$$\mathrm{Log}_{2\pi}(z) = \log_e|z| + i\mathrm{Arg}_{2\pi}(z) = \log_e|z| + i\mathrm{Arg}_{3\pi}(z) = \mathrm{Log}_{3\pi}(z)$$

$\mathrm{Log}_{2\pi}(z)$ and $\mathrm{Log}_{3\pi}(z)$ have the domains $\mathbb{C}_{2\pi}$ and $\mathbb{C}_{3\pi}$ respectively. Since $V \subset \mathbb{C}_{2\pi}$ and $\mathbb{C}_{3\pi}$ and

$\mathrm{Log}_{2\pi}(z) = \mathrm{Log}_{3\pi}(z)$ when $z \in V$

then $\mathrm{Log}_{2\pi}$ and $\mathrm{Log}_{3\pi}$ are direct analytic continuations of each other. (see Unit C3, Section 1, Para. 1)

Therefore $(f_1, \mathbb{C}_\pi)$, $(f_1, \mathbb{C}_{2\pi})$, and $(f_1, \mathbb{C}_{3\pi})$ form a chain (Unit C3, Section 2, Para. 3), and since $\mathbb{C}_\pi = \mathbb{C}_{3\pi}$ then it is a closed chain.

As $f_1(1) = \mathrm{Log}_\pi(1) = \log_e|1| + i\mathrm{Arg}_\pi(1) = 0 + i0 = 0$, and

$f_3(1) = \mathrm{Log}_{3\pi}(1) = \log_e|1| + i\mathrm{Arg}_{3\pi}(1) = 0 + i2\pi = 2\pi i$,

then $f_1 \neq f_3$.

2002 Question 7

(a) 1 mark

q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C}$ and the conjugate velocity function $\overline{q}(z) = -iz$ is analytic on $\mathbb{C}$**.** Therefore q is a model fluid flow on $\mathbb{C}$ (Unit D2, Section 1, Para. 14).

(b) 6 marks

The complex potential function Ω is a primitive of $\overline{q}(z)$ (Unit D2, Section 2, Para. 1). Therefore the complex potential function $\Omega(z) = -iz^2/2$ and the stream function

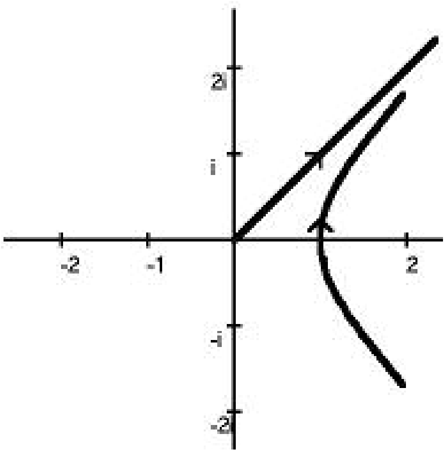
$$\Psi(x, y) = \operatorname{Im}\Omega(z) \qquad \text{(Unit D2, Section 4, Para. 4)}$$

$$= \operatorname{Im}\left(-\tfrac{i}{2}(x+iy)^2\right) \qquad \text{, where } z = x + iy$$

$$= \operatorname{Im}\left(-\tfrac{i}{2}\left(x^2 - y^2 + 2ixy\right)\right) = \tfrac{1}{2}\left(y^2 - x^2\right)$$

A streamline through 1 is given by $\tfrac{1}{2}\left(y^2 - x^2\right) = \Psi(1,0) = -\tfrac{1}{2}$.
Therefore the streamline through i has the equation $y^2 = x^2 - 1$.
At 1 the velocity function q(1) = i (positive y direction)

A streamline through 1 + i is given by $\tfrac{1}{2}\left(y^2 - x^2\right) = \Psi(1,1) = 0$ or $y^2 = x^2$.
Since the streamline goes through 1 + i we must have y = x..
At 1 + i the velocity function q(1 + i) = i(1 - i) = 1 + i (north-east)
At 0 the velocity function q(0) = 0.



(c) 1 mark

Since q is a model flow on $\mathbb{C}$ then the integral $\int_\Gamma \overline{q}(z)dz = 0$ for each simple-closed contour surrounding 0 (Unit D2, Section 1, Para. 13). Therefore 0 is neither a source or a vortex (Unit D1, Section 1, Para. 15).

2002 Question 8

(a) 3 marks [Unit D3, Exercise 1.2 (b)]

If α is a fixed point (Unit D3, Section 1, Para. 3) then

$f(\alpha) = 2\alpha(1 - \alpha) = \alpha$.

Since $\alpha(1 - 2\alpha) = 0$ then the fixed points are at $\alpha = 0$ and $\alpha = ½$.

$f^{/}(z) = 2 - 4z$.

When $z = 0$ then $| f^{/}(z) | = 2$. Therefore 0 is a repelling fixed point (Unit D3 Section 1, Para. 5).

When $z = ½$ then $| f^{/}(z) | = 0$. Therefore ½ is a super-attracting fixed point.

(b) 5 marks

(b)(i)

$P_c(0) = \frac{1}{2}(-3+i)$.

$P_c^2(0) = \frac{1}{4}(-3+i)^2 + \frac{1}{2}(-3+i) = (2 - \frac{3}{2}i) + \frac{1}{2}(-3+i) = \frac{1}{2} - i$.

$P_c^2(0) = (\frac{1}{2} - i)^2 + \frac{1}{2}(-3+i) = (-\frac{3}{4} - i) + \frac{1}{2}(-3+i) = -\frac{9}{4} - \frac{1}{2}i$.

As $\left| P_c^3(0) \right| > 2$ then c does not lie in the Mandelbrot set (Unit D3, Section 4, Para. 5).

(b)(ii) $|c|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Hence $\left(8|c|^2 - \frac{3}{2}\right)^2 + 8\,\mathrm{Re}\,c = \left(\frac{5}{2}\right)^2 + 8\left(-\frac{1}{2}\right) = \frac{25}{4} - 4 = \frac{9}{4} < 3$.

Therefore P_c has an attracting fixed point (Unit D3, Section 4, Para. 9). Hence c belongs to the Mandelbrot set (Unit D3, Section 4, Para. 8).

2002 Question 9

(a) 8 marks

(a)(i)

Putting z = x + iy we have

f(x + iy) = sin(x – iy)

= sin x cos(iy) – cos x sin(iy) (Unit A2, Section 4, Para. 5)

= sin x cosh y – i cos x sinh y (Unit A2, Section 4, Para. 7)

= u(x, y) + i v(x, y)

where u(x,y) = sin x cosh y, and v(x,y) = - cos x sinh y.

(a)(ii)

$$\frac{\partial u}{\partial x}(x,y) = \cos x * \cosh y\,, \qquad \frac{\partial u}{\partial y}(x,y) = \sin x * \sinh y\,,$$
$$\frac{\partial v}{\partial x}(x,y) = \sin x * \sinh y\,, \qquad \frac{\partial v}{\partial y}(x,y) = -\cos x * \cosh y$$

If f is differentiable the Cauchy-Riemann equations (Unit A4, Section 2, Para. 1) hold.

They will hold at (a, b) if

$$\frac{\partial u}{\partial x}(a,b) = \cos a * \cosh b = -\cos a * \cosh b = \frac{\partial v}{\partial y}(a,b), \text{ and}$$
$$\frac{\partial v}{\partial x}(a,b) = \sin a * \sinh b = -\sin a * \sinh b = -\frac{\partial u}{\partial y}(a,b)$$

For real x, cosh x > 0. Therefore to satisfy the 1st condition we must have cos a = 0. To also satisfy the 2nd equation then, as sin a = 1, we must have sinh b = 0 and hence b = 0. Therefore both equations are satisfied when

$$z \in \{ (n + ½)\pi : n \in \mathbb{Z}\} = A.$$

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous at each z ∈ A.
3. satisfy the Cauchy-Riemann equations at each z ∈ A.

then, by the Cauchy-Riemann Converse Theorem (Unit A4, Section 2, Para. 3), f is differentiable on A.

(b) 10 marks

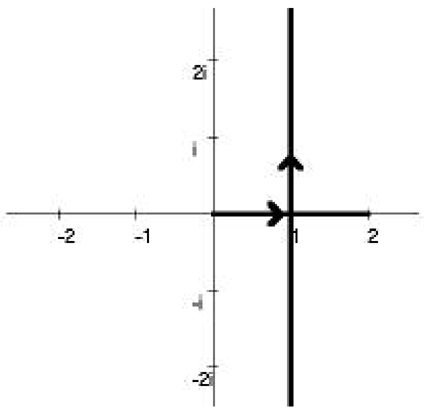
(b)(i) g(z) is analytic on the region $\mathbb{C} - \{i\}$ (Unit A4, Section 3, Para. 4), and

$$g'(z) = -\frac{1}{(z-i)^2} \text{ on } \mathbb{C} - \{i\}.$$

On the region $\mathbb{C}$ - {i} since $g'(z) \neq 0$ and g is analytic, then g is also conformal on this region (Unit A4, Section 4, Para. 6).

(b)(ii) As ½ is in the domain of γ_1 we have $\gamma_1\left(\frac{1}{2}\right) = 1$.

As 0 is in the domain of γ_2 we have $\gamma_2(0) = 1$. Therefore Γ_1 and Γ_2 meet at the point 1.



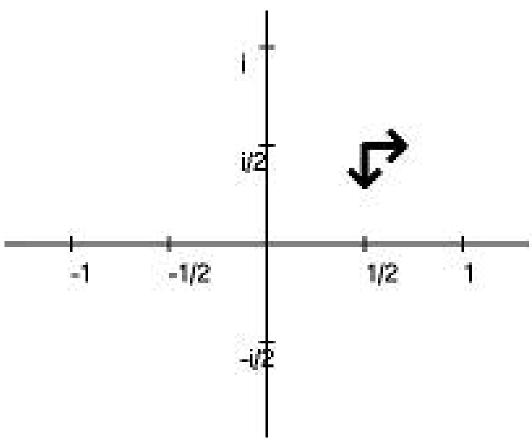
(b)(iii)

As g is analytic on $\mathbb{C}$ - {i} and $g'(i) \neq 0$ then a small disc centred at 1 is mapped approximately to a small disk centred at

g(1) = 1/(1 – i) = (1 + i)/2 (Unit A4, Section 1, Para. 11).

The disc is rotated by Arg $g'(1)$ = Arg(-1/(1 - i)^2) = Arg(1/2i) = -π/2, and scaled by the factor | $g'(1)$| = ½.

In the diagram below g(Γ_2) is the horizontal line (Unit A4, Section 4, Para. 4)



2002 Question 10

(a) 8 marks

(a)(i) $z^3 - 2z^2 + z = z(z^2 - 2z + 1) = z\,(z - 1)^2$.

Therefore f has a simple pole at $z = 0$ and a pole of order 2 at $z = 1$ (Unit B4, Section 1, Para. 3).

(a)(ii) $f(z) = \dfrac{1}{(z-1)^2}\left\{\dfrac{1}{1+(z-1)}\right\}$, when $z \in \mathbb{C} - \{0, 1\}$

$$= \frac{1}{(z-1)^2}\sum_{n=0}^{\infty}(-1)^n (z-1)^n \text{, when } 0 < |z-1| < 1 \qquad \text{(Unit B3, Section 3, Para. 5)}$$

$$= \sum_{n=0}^{\infty}(-1)^n (z-1)^{n-2}$$

Therefore the Laurent series about 1 for f on $\{z: 0 < |z - 1| < 1\}$ is

$$\frac{1}{(z-1)^2} - \frac{1}{z-1} + 1 - (z-1) + (z-1)^2 - \ldots + (-1)^n (z-1)^{n-2} + \ldots$$

(b) 10 marks

(b)(i) By the Composition Rule (Unit B3, Section 4, Para. 3) the Taylor series for g about 0 on $\mathbb{C}$ is

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left\{\sum_{m=0}^{\infty}(-1)^m \frac{z^{2m+1}}{(2m+1)!}\right\}^{2n}$$

$$= 1 - \frac{1}{2!}\left\{z - \frac{z^3}{3!} + \ldots\right\}^2 + \frac{1}{4!}\left\{z - \frac{z^3}{3!} + \ldots\right\}^4 - \ldots$$

$$= 1 - \left\{\frac{z^2}{2} - \frac{z^4}{6} + \ldots\right\} + \left\{\frac{z^4}{24} - \ldots\right\} = 1 - \frac{z^2}{2} + \frac{5z^4}{24} - \ldots \quad \text{up to the term in } z^4.$$

Since g is analytic on $\mathbb{C}$ then by Taylor's Theorem (Unit B3, Section 3, Para. 1) then the representation of g is unique on all open discs centred at 0 in the sense that if

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

then the coefficients a_n are those found above.

$z\,g(1/z)$ is analytic on the punctured disc $\mathbb{C}$ - {0}.

The Laurent series about 0 for $z\,g(1/z)$ on this disc is

$$z\left(1 - \frac{1}{2z^2} + \frac{5}{24z^4} - \ldots\right) = z - \frac{1}{2z} + \frac{5}{24z^3} - \ldots = \sum_{n=-\infty}^{\infty} a_n z^n$$

Therefore as C is a circle with centre 0 (Unit B4, Section 4, Para. 2)

$$\int_C z g\left(\frac{1}{z}\right) dz = 2\pi i a_{-1} = 2\pi i\left(-\frac{1}{2}\right) = -\pi i$$

$z^2\,g(1/z)$ is analytic on the punctured disc $\mathbb{C}$ - {0}.

The Laurent series about 0 for $z^2\,g(1/z)$ on this disc is

$$z^2\left(1 - \frac{1}{2z^2} + \frac{5}{24z^4} - \ldots\right) = z^2 - \frac{1}{2} + \frac{5}{24z^2} - \ldots = \sum_{n=-\infty}^{\infty} a_n z^n$$

Therefore as C is a circle with centre 0 (Unit B4, Section 4, Para. 2)

$$\int_C z^2 g\left(\frac{1}{z}\right) dz = 2\pi i a_{-1} = 0$$

2002 Question 11

(a) 10 marks

(a)(i)

By the Triangle Inequality (Unit A1, Section 5, Para. 2a)

$$
\begin{aligned}
| e^z + 5 | &\le | e^z | + 5 \\
&= e^{\operatorname{Re} z} + 5 && \text{(Unit A2, Section 4, Para. 2b)} \\
&\le e^2 + 5 && \text{when } |z| = 2. \\
&< 3^2 + 5 = 14 && \text{since } e < 3.
\end{aligned}
$$

(a)(ii)

Let $f(z) = e^z + z^4 + 5$. Then f is analytic on the simply-connected region $\mathbb{C}$.

Let $\Gamma = \{ z : |z| = 1\}$ and $g(z) = 5$.
When $z \in \Gamma$ we have

$$
\begin{aligned}
| f(z) - g(z) | &= | e^z + z^4 | \\
&\le | e^z | + | z^4 | && \text{Triangle inequality} \\
&= e^{\operatorname{Re} z} + | z^4 | \\
&\le e^1 + 1 \\
&< 5 = g(z).
\end{aligned}
$$

As f and g are analytic (Unit A4, Section 1, Para. 7) on the simply-connected region $\mathbb{C}$, and Γ is a simple-closed contour in $\mathbb{C}$ then by Rouché's Theorem (Unit C2, Section 2, Para. 4) f has the same number of zeros inside Γ as g.
Therefore $f(z) = 0$ has no solutions inside the disc $\{ z : |z| < 1\}$.

By the Backwards form of the triangle inequality when $|z| = 1$ (Unit A1, Section 5, Para. 2b)

$$
\begin{aligned}
| 5 + e^z + z^4 | &\ge | |5| - | e^z + z^4 | | \\
&\ge | 5 - (e + 1) | \\
&> 0.
\end{aligned}
$$

Therefore there are no zeros on $\{z : |z| = 1\}$.

Let $\Gamma = \{ z : |z| = 2\}$ and $g(z) = z^4$.
When $z \in \Gamma$ we have

$$
\begin{aligned}
| f(z) - g(z) | &= | e^z + 5 | \\
&\le 14 && \text{using part (a)(i).} \\
&< | z^4 | = 16 = g(z).
\end{aligned}
$$

As f and g are analytic (Unit A4, Section 1, Para. 7) on the simply-connected region $\mathbb{C}$, and Γ is a simple-closed contour in $\mathbb{C}$ then by Rouché's Theorem (Unit C2, Section 2, Para. 4) f has the same number of zeros inside Γ as g.
Therefore $f(z) = 0$ has 4 solutions inside the disc $\{ z : |z| < 2\}$.

Therefore 4 solutions of f(z) = 0 lie in the annulus { z : 1 < |z| < 2}.
(a) 8 marks

I shall use the result in Unit C1, Section 3, Para. 9.

Let p(t) = 1 – t, q(t) = $(1 + t^2)$ t, $r(t) = \frac{p(t)}{q(t)}\exp(ikt)$, where k = 1.

Since (1) the degree of q exceeds that of p by more than 1.
(2) the only pole of p/q on the real axis (at 0) is simple,

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\exp(ikt)\,dt = 2\pi iS + \pi iT$$

where S is the sum of the residues of the function r(z) at those poles in the upper half-plane, and T is the sum of the residues of the function r(z) at those poles on the real axis.

The only pole in the upper half-plane is at z = i and

$$S = \text{Res}(r, i) = \lim_{z \to i} (z-i)\,r(z)$$

$$= \frac{(1-i)\exp(i^2)}{(i+i)i} = \frac{(1-i)\,e^{-1}}{-2} = -\frac{(1-i)}{2e}.$$

$$T = \text{Res}(r, 0) = \lim_{z \to 0} (z-0)\,r(z) = \frac{1}{1}e^0 = 1.$$

Hence $\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\exp(ikt)\,dt = 2\pi i\left\{-\frac{1-i}{2e}\right\} + \pi i(1)$.

Therefore taking the real part gives

$$\int_{-\infty}^{\infty} \frac{1-t}{1+t^2}\frac{\cos t}{t}\,dt = -\frac{\pi}{e}.$$

2002 Question 12

(a) 6 marks

(a)(i) **True**.

$\frac{1}{z} = \frac{az+b}{cz+d}$ where a = 0, b = 1, c = 1, and d = 0.

a, b, c, d $\in \mathbb{C}$, and ad - bc = -1 $\neq$ 0 so 1/z is a Möbius transformation (Unit D1, Section 1, Para. 1).

(a)(ii) **False**.

Selecting two different sets of 3 points on the boundary of the half-plane and mapping them to the same 3 points on the boundary of the open unit disc will give different Möbius transformations.

(a)(iii) **False**.

If there was such a transformation f then f would be an entire function.
Also f is bounded since |f(z)| < 1.
By Liouville's theorem (Unit B2, Section 2, Para. 2) then f must be a constant function. Therefore there is no such transformation.

(a) 12 marks

(b)(i)

The boundary of the open half-plane R on $\hat{\mathbb{C}}$ is the extended line which has inverse points 1 and –1.

f(1) = 0, f(-1) = ∞, f(0) = -1.

The inverse points are mapped to the inverse points of the unit disc D (Unit D1, Section 3, Para. 6), and the boundary point 0 of R is mapped to the boundary point of the unit disc D (Unit D1, Section 4 Para. 3). As 0 and ∞ are inverse points the 0 is the centre of the circle (Unit D1, Section 3, Para. 5). As -1 is on the circle it has radius 1.

Therefore the mapping of these 3 points shows that f maps the half-plane onto the unit disc D.

(b)(ii)

f maps the extended real axis to a generalized circle.

As f(-1) = ∞, f(0) = -1, and f(1) = 0 then the extended real axis is mapped to the extended real axis.

Therefore the real axis is mapped to the real axis excluding the point (1, 0).

(b)(iii)

The principal square root function

$$h(z) = \sqrt{z} \qquad z \in \mathbb{C} - \{ x \in \mathbb{R} : x \le 0\}$$

is a conformal mapping (Unit A4, Section 4, Para. 6) from $\mathbb{C} - \{ x \in \mathbb{R} : x \le 0\}$ onto *R* as $h'(z) = 1/\sqrt{z} \neq 0$ on its domain.

Therefore a conformal mapping from $\mathbb{C} - \{ x \in \mathbb{R} : x \le 0\}$ to *D* is $f \circ h$.

$$g(z) = (f \circ h)(z) = \frac{\sqrt{z} - 1}{\sqrt{z} + 1}.$$

(b)(iv)

$$g^{-1} = (f \circ h)^{-1} = (h^{-1} \circ f^{-1})$$

Therefore $g^{-1}(z) = \left(\dfrac{z+1}{-z+1}\right)^2$.