1998 Question 1

(a) 5 marks

(a)(i) $w = \dfrac{1}{1+i} = \dfrac{1}{1+i}\left(\dfrac{1-i}{1-i}\right) = \dfrac{1-i}{2}$

Therefore Arg w = - π / 4. (Unit A1, Section 2, Para. 8)

(a)(ii) $|w| = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

The principal fourth root of w is (Unit A1, Section 3, Para. 3)

$$z_0 = 2^{-1/8}\left(\cos\left(\frac{1}{4}\left(-\frac{\pi}{4}\right)\right) + i\sin\left(\frac{1}{4}\left(-\frac{\pi}{4}\right)\right)\right)$$

$$= 2^{-1/8}\left(\cos\left(-\frac{\pi}{16}\right) + i\sin\left(-\frac{\pi}{16}\right)\right)$$ (Unit A1, Section 2, Para. 6)

(b) 3 marks

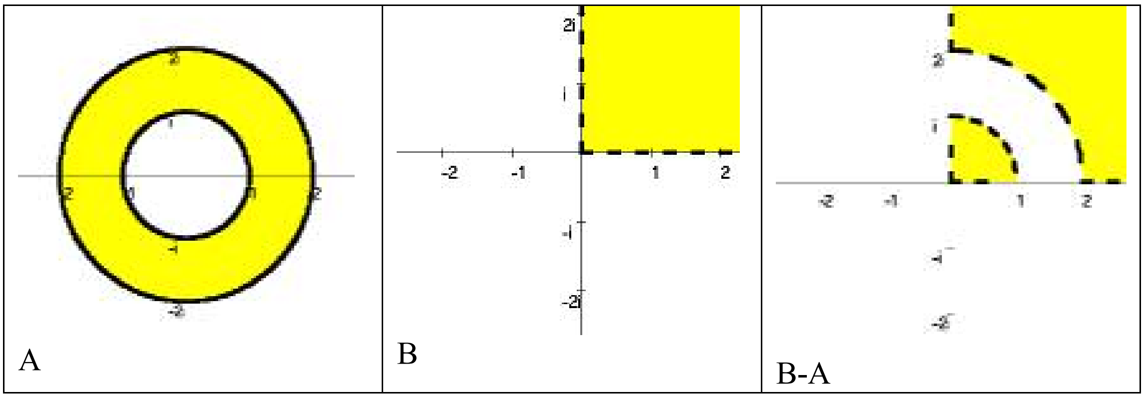
$$(-1)^{3i} = \exp(3i\,\text{Log}(-1))$$ (Unit A2, Section 5, Para. 3)

$$= \exp(3i(\log_e|-1| + i\text{Arg}(-1)))$$ (Unit A2, Section 5, Para. 1)

$$= \exp(3i(0 + i\pi)) = \exp(-3\pi).$$

1998 Question 2

(a) 3 marks



(b) 5 marks

(b)(i) (Unit A3, Section 4, Paras. 6 and 7)

A is not a region since it is not open.
B is a region.
B-A is not a region as it is not connected.

(b)(ii) (Unit A3, Section 5, Para. 5)

A is compact.
B is not compact as it is not closed or bounded.
B - A is not compact as it is not closed or bounded.

1998 Question 3

(a) 4 marks

The standard parametrization (Unit A2, Section 2, Para. 3) for Γ_1 is

$\gamma_1(t) = (1 - t) + ti, \qquad t \in [0, 1]$

and $\gamma_1'(t) = -1 + i$.

As γ_1 is differentiable on [0, 1], γ_1' is continuous on [0, 1], and γ_1' is non-zero on [0, 1] then γ_1 is a smooth path (Unit A4, Section 4, Para. 3).

Since γ_1 is a smooth path then (Unit B1, Section 2, Para. 1)

$$\int_{\Gamma_1} \operatorname{Re} z \, dz = \int_0^1 \{\operatorname{Re} \gamma_1(t)\} \gamma_1'(t) \, dt$$

$$= \int_0^1 (1-t)(-1+i) \, dt$$

$$= (-1+i)\left[-\frac{(1-t)^2}{2}\right]_0^1$$

$$= \frac{-1+i}{2}$$

(b) 3 marks

Let $f(z) = 1/z$, $F(z) = \operatorname{Log} z$ and the region $\boldsymbol{R} = \mathbb{C} - \{x : x \le 0\}$.

f is continuous on $\boldsymbol{R}$ and F is a primitive of f on $\boldsymbol{R}$ (Unit A4, Section 3, Para. 4). Thus, by the Fundamental Theorem of Calculus (Unit B1, Section 3, Para. 2), since Γ_1 is a contour in $\boldsymbol{R}$

$$\int_{\Gamma_1} \frac{1}{z} \, dz = F(i) - F(1)$$

$$= \operatorname{Log} i - \operatorname{Log} 1$$

$$= \{\log_e |i| + i \operatorname{Arg} i\} - 0 \qquad \text{(Unit A2, Section 5, Para. 1)}$$

$$= \frac{i\pi}{2}$$

(c) 1 mark

Since Γ_2 is also a contour in $\boldsymbol{R}$ with the same start and end points as Γ_1 then by the Contour Independence Theorem (Unit B1, Section 3, Para. 4)

$$\int_{\Gamma_2} \frac{1}{z} \, dz = \int_{\Gamma_1} \frac{1}{z} \, dz = \frac{i\pi}{2}$$

1998 Question 4

(a) 2 marks

The zeros of $z^2 + 2$ are at $z = \pm 2^{1/2}\, i$. These are outside the contour C_1.

R = {z : |z| < 2} is a simply-connected region (Unit B2, Section 1, Para. 3) and $\frac{z^3}{z^2+2}$ is analytic (quotient rule) on ***R***. Since C_1 is a closed contour in ***R*** then by Cauchy's Theorem (Unit B2, Section 1, Para. 4)

$$\int_{C_1} \frac{z^3}{z^2+2}\,dz = 0$$

(b) 3 marks

[Follow similar strategy to that given in Unit B2, Section 4, Para. 3]

Both the zeros of $z^2 + 2$ are inside the contour C_2.

$$\frac{1}{z^2+2} = \frac{1}{\left(z - i\sqrt{2}\right)\left(z + i\sqrt{2}\right)} = \frac{1}{2i\sqrt{2}}\left\{\frac{1}{z - i\sqrt{2}} - \frac{1}{z + i\sqrt{2}}\right\}$$

Therefore $\int_{C_2} \frac{z^3}{z^2+2}\,dz = \frac{1}{2i\sqrt{2}}\left\{\int_{C_2} \frac{z^3}{z - i\sqrt{2}}\,dz - \int_{C_2} \frac{z^3}{z + i\sqrt{2}}\,dz\right\}$

As $\mathbb{C}$ is a simply-connected region which contains the simple-closed contour C_2 (Unit B2, Section 1, Para. 1) and $f(z) = z^3$ is analytic on $\mathbb{C}$ then using Cauchy's Integral Formula (Unit B2, Section 2, Para. 1) gives

$$\int_{C_2} \frac{z^3}{z^2+2}\,dz = \frac{1}{2i\sqrt{2}}\left\{2\pi i\, f\left(i\sqrt{2}\right) - 2\pi i\, f\left(-i\sqrt{2}\right)\right\}$$

$$= \frac{\pi}{\sqrt{2}}\left\{-2\sqrt{2}i - 2\sqrt{2}i\right\} = -4\pi i$$

(c) 3 marks

C_2 is a simply-closed contour in the simply-connected region $\mathbb{C}$ and $f(z) = z^3$ is analytic on $\mathbb{C}$.

Since the point –2 lies in C_2 then by Cauchy's n'th Derivative formula (Unit B2, Section 3, Para. 1), with n = 1 and $\alpha = -2$, we have

$$\int_{C_2} \frac{z^3}{(z+2)^2}\,dz = \frac{2\pi i}{1!} f'(-2) = 2\pi i * 3(-2)^2 = 24\pi i$$

1998 Question 5

Identical to 2001 Question 5.

1998 Question 6

8 marks

f and g are not direct analytic continuations of each other since $D_0 \cap D_1 = \phi$.

$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^2$ is a geometric series with sum $\frac{1}{1-\frac{z}{2}} = \frac{2}{2-z}$ on D_0.

$\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^2$ is a geometric series with sum $\frac{1}{1-\frac{2}{z}} = \frac{z}{z-2}$ on D_1.

Therefore $g(z) = -\sum_{n=1}^{\infty} \left(\frac{2}{z}\right)^2 = -\frac{2}{z}\sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^2 = -\frac{2}{z}\left(\frac{z}{z-2}\right) = \frac{2}{2-z}$ on D_1.

Let $h(z) = \frac{2}{2-z}$ on $\mathbb{C}$ - {2}.

f and h are analytic functions whose domains are regions and f(z) = h(z)
for $z \in D_0 \subseteq D_0 \cap (\mathbb{C} - \{2\})$ then h(z) is an analytic continuation of f(z) to $\mathbb{C} - \{2\}$ (Unit C3, Section 1, Para. 1).

Since the domains of h and g are regions and h(z) = g(z) for $z \in D_1 \subseteq (\mathbb{C} - \{2\}) \cap D_1$ then g(z) is an analytic extension of h(z) to D_1.

As the functions (f, D_0), $(h, \mathbb{C} - \{2\})$, (g, D_1) form a chain then f and g are indirect analytic continuations of each other (Unit C3, Section 2, Para. 3).

1998 Question 7

(a) 1 mark

The conjugate velocity function $\overline{q}(z) = 1/z^2$.
Since q is a steady continuous 2-dimensional velocity function on the region $\mathbb{C} - \{0\}$ and $\overline{q}$ is analytic on $\mathbb{C} - \{0\}$ then q is a model fluid flow (Unit D2, Section 1, Para. 14).

(b) 5 marks

On $\mathbb{C} - \{0\}$, $\Omega(z) = -\frac{1}{z}$ is a primitive of $\overline{q}$. Therefore Ω is a complex potential function for the flow (Unit D2, Section 2, Para. 1).

The stream function $\Psi(x, y) = \operatorname{Im}\Omega(z)$ (Unit D2, Section 2, Para. 4)

$$= \operatorname{Im}\left(-\frac{1}{x+iy}\right) \quad \text{, where } z = x + iy,\ (x,y) \neq (0, 0)$$

$$= \operatorname{Im}\left(-\frac{x-iy}{x^2+y^2}\right) = \frac{y}{x^2+y^2}$$

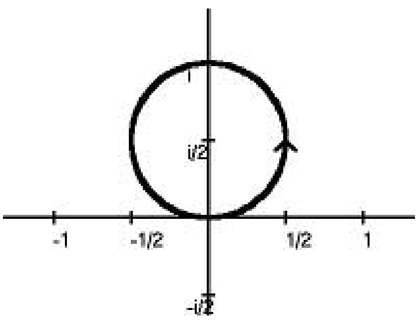
A streamline through the point i satisfies the equation

$$\frac{y}{x^2+y^2} = \Psi(0,1) = 1 \quad \text{(Unit D2, Section 2, Para. 4)}$$

Therefore the streamline through i has the equation $x^2 + y^2 - y = 0$ or

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

Since q(i) = -1 (-x direction) then the direction of flow is as shown.



(c) 2 marks

The flux of q across the unit circle C = {z : |z| = 1}is (Unit D2, Section 1, Para. 10)

$$\operatorname{Im}\left(\int_C \overline{q}(z)dz\right) = \operatorname{Im}\left(\int_C \frac{1}{z^2}dz\right) = 0 \qquad \text{by Cauchy's Residue Theorem.}$$

1998 Question 8

(a) 3 marks

Same as 2002 Qu. 8(a).

(b) 5 marks

(b)(i) [Unit D3, Exercise 4.1(c)]

$P_c(0) = -1 + i$.
$P_c^2(0) = (-1+i)^2 + (-1+i) = -2i + (-1+i) = -1-i.$
$P_c^3(0) = (-1-i)^2 + (-1+i) = 2i + (-1+i) = -1+3i.$

As $\left| P_c^3(0) \right| > 2$ then c does not lie in the Mandelbrot set (Unit D3, Section 4, Para. 5).

(b)(ii)

Since $\left| c+1 \right| = \left| -\frac{1}{8}i \right| = \frac{1}{8} < \frac{1}{4}$ then P_c has an attracting 2-cycle (Unit D3, Section 4, Para. 9). Therefore c belongs to the Mandelbrot set (Unit D3, Section 4, Para. 8).

1998 Question 9

(a) 8 marks

Putting z = x + iy we have

$$f(z) = \bar{z} - |z|^2 = (x - iy) - (x^2 + y^2) = u(x, y) + iv(x, y),$$

where $u(x,y) = x - x^2 - y^2$, and $v(x,y) = -y$.

$$\frac{\partial u}{\partial x}(x, y) = 1 - 2x, \quad \frac{\partial u}{\partial y}(x, y) = -2y \quad \frac{\partial v}{\partial x}(x, y) = 0, \quad \frac{\partial v}{\partial y}(x, y) = -1$$

If f is differentiable the Cauchy-Riemann equations (Unit A4, Section 2, Para. 1) hold.

They will hold at (a, b) if

$$\frac{\partial u}{\partial x}(a, b) = 1 - 2a = -1 = \frac{\partial v}{\partial y}(a, b), \text{ and}$$

$$\frac{\partial v}{\partial x}(a, b) = 0 = 2b = -\frac{\partial u}{\partial y}(a, b)$$

Since the Cauchy-Riemann equations only hold at (1, 0) then

f is not differentiable on $\mathbb{C} - \{1\}$.

As f is defined on the region $\mathbb{C}$, and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

1. exist on $\mathbb{C}$
2. are continuous at (1, 0).
3. satisfy the Cauchy-Riemann equations at (1, 0)

then, by the Cauchy-Riemann Converse Theorem (Unit A4, Section 2, Para. 3), f is differentiable at (1, 0).

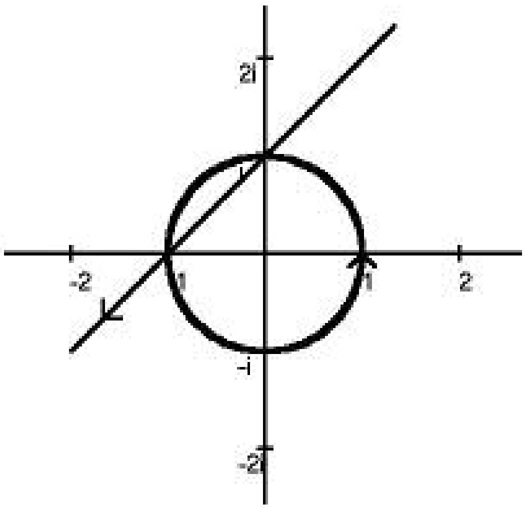
Therefore f is only differentiable at (1, 0).

(b) 10 marks

(i) g(z) is analytic on the region $\mathbb{C} - \{0\}$ (Unit A4, Section 3, Para. 4), and $g'(z) = -\frac{2}{z^3}$ on $\mathbb{C} - \{0\}$.

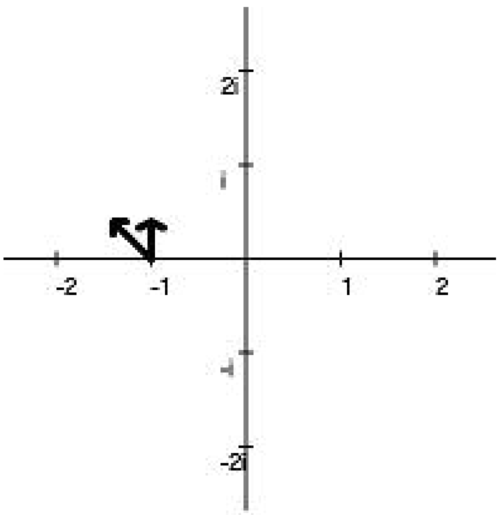
As $g'(i) = -\frac{2}{i^3} = -2i \neq 0$ and g is analytic at i , then g is conformal at z = i. (Unit A4, Section 4, Para. 6)

(ii) $\pi/2$ is in the domain of γ_1 so $\gamma_1(\pi/2) = e^{i\pi/2} = i$.
0 is in the domain of γ_2 so $\gamma_2(0) = i$. Therefore Γ_1 and Γ_2 meet at the point i.



(iii) As g is analytic on $\mathbb{C}$- {0} and $g'(i) \neq 0$ then a small disc centred at i is mapped approximately (Unit A4, Section 1, Para. 11) to a small disc centred at g(i) = -1. The disc is rotated by Arg ($g'(i)$) = Arg -2i = $-\pi/2$, and scaled by a factor | $g'(i)$| = 2.

In the diagram below g(Γ_1) is the vertical line.



1998 Question 10

(a) 9 marks

(i) There are poles at z = 0, and z = 3. As $\lim_{z \to 0} (z-0)f(z) = -\frac{1}{3}$ and $\lim_{z \to 3} (z-3)f(z) = \frac{1}{3}$ then these are simple poles.

(ii) Let z = 1 + h. For $z \neq 0, 3$ we have

$$f(z) = \frac{1}{(1+h)(h-2)} = -\frac{1}{3}\left\{\frac{1}{2-h} + \frac{1}{1+h}\right\}$$

$$= -\frac{1}{3}\left\{\frac{1}{2(1-h/2)} + \frac{1}{h(1+1/h)}\right\}$$

As $1 < |z-1| = |h| < 2$ then $|h/2| < 1$ and $|1/h| < 1$. Therefore

$$f(z) = -\frac{1}{6}\left\{1 + \left(\frac{h}{2}\right) + \left(\frac{h}{2}\right)^2 + \ldots\right\} - \frac{1}{3h}\left\{1 + \left(-\frac{1}{h}\right) + \left(-\frac{1}{h}\right)^2 + \ldots\right\}$$

when $1 < |h| < 2$. (Unit B3, Section 3, Para. 5)

$$= \ldots + \frac{1}{3(z-1)^2} - \frac{1}{3(z-1)} - \frac{1}{6} - \frac{(z-1)}{12} - \frac{(z-1)^2}{24} + \ldots$$

when $1 < |z-1| < 2$.

(b) 9 marks

(i)

$$\cos z - 1 = -\frac{z^2}{2!} + \frac{z^4}{4!} - \dots, \qquad \text{for } z \in \mathbb{C}. \text{ (Unit B3, Section 3, Para. 5)}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \dots \quad , \qquad \text{for } z \in \mathbb{C}. \text{ (Unit B3, Section 3, Para. 5)}$$

By the Composition Rule (Unit B3, Section 4, Para. 3)

$$g(z) = 1 + \left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \dots \right) + \frac{1}{2!}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \dots \right)^2 + \dots$$

$$= 1 - \frac{z^2}{2} + z^4\left(\frac{1}{24} + \frac{1}{8} \right) + \dots = 1 - \frac{z^2}{2} + \frac{z^4}{6} + \dots \text{ for } z \in \mathbb{C}.$$

Since g is analytic on $\mathbb{C}$ then by Taylor's Theorem (Unit B3, Section 3, Para. 1) this is the unique power series for g on $\mathbb{C}$.

(ii)

$z^3\, g(1/z)$ is analytic on the punctured disc $\mathbb{C} - \{0\}$.

The Laurent series about 0 for $z^3\, g(1/z)$ on this disc is

$$z^3 g\left(\frac{1}{z}\right) = z^3 - \frac{z}{2} + \frac{1}{6z} + \dots = \sum_{n=-\infty}^{\infty} a_n z^n$$

Therefore as C is a circle with centre 0 (Unit B4, Section 4, Para. 2)

$$\int_C z^3 g\left(\frac{1}{z}\right) dz = 2\pi i a_{-1} = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}$$

1998 Question 11

(a) 8 marks

(a)(i)

Let $g(z) = 3z^2$ and $f(z) = z^3 + 3z^2 + 1$.

Using the Triangle Inequality (Unit A1, Section 5, Para. 2) then on $\Gamma = \{ z : |z| = 1\}$ we have

$$| f(z) - g(z) | = | z^3 + 1 | \le | z^3 | + 1 = 2$$
$$< 3 = | 3z^2 | = | g(z) |$$

As f and g are analytic (Unit A4, Section 1, Para. 7) on the simply-connected region $\mathbb{C}$, and Γ is a simple-closed contour in $\mathbb{C}$ then by Rouché's Theorem (Unit C2, Section 2, Para. 4) f has the same number of zeros inside Γ as g.
Therefore $f(z) = 0$ has 2 solutions in the disc $\{ z : |z| < 1\}$.

Let $g(z) = z^3$ and $\Gamma = \{ z : |z| = 4\}$.
Using the Triangle Inequality then on Γ we have

$$| f(z) - g(z) | = | 3z^2 + 1 | \le | 3z^2 | + 1 = 49$$
$$< 64 = | z^3 | = | g(z) |$$

As f and g are analytic on the simply-connected region $\mathbb{C}$, and Γ is a simple-closed contour in $\mathbb{C}$ then by Rouché's Theorem f has the same number of zeros inside Γ as g. Therefore $f(z) = 0$ has 3 solutions in the disc $\{ z : |z| < 4\}$.

On $\{z : |z| = 1\}$ then (Unit A1, Section 5, Para. 3e)

$$|f(z)| = | 3z^2 + z^3 +1 | \ge |3z^2| - | z^3| - 1 = 3 - 1 - 1 > 0.$$

As there are no zeros on $\{z : |z| = 1\}$ then there is a single solution on the annulus $\{z : 1 < |z| < 4\}$.

(a)(ii) Taking the conjugate of $f(z) = 0$ gives

$$(\overline{z})^3 + 3(\overline{z})^2 + 1 = 0$$

Therefore if z is a solution so is its conjugate. As there is only one solution in the annulus then it it must be real. If z is real then $3z^2 + 1 > 0$ so if $f(z) = 0$ then z^3 is negative.

Therefore the root in the annulus is real and negative.

(b) 10 marks

(b)(i)

Let $f(z) = \exp(z^3)$ and $R = \{z : |z| < 3\}$.

If we write $z = r\, e^{i\theta}$ then $z^3 = r^3\, e^{i3\theta}$ and

$$|\exp(z^3)| = \exp(\operatorname{Re} z^3) = \exp(r^3 \cos 3\theta).$$

Since f is analytic on the bounded region R and continuous on R then, by the Maximum Principle, there exists an $\alpha \in \partial R$ such that $|f(z)| \le |f(\alpha)|$ for all $z \in \overline{R}$ (Unit C2, Section 4, Para. 4).

On ∂R each point can be written in the form $z = 3e^{i\theta}$ for $\theta \in [0, 2\pi]$.

Therefore $\max\{|\exp(z^3)| : z \in \partial R\}$

$$= \max\{|\exp(27\cos 3\theta)| : \theta \in [0, 2\pi]\}$$
$$= e^{27}.$$

The maximum occurs when $\cos 3\theta = 1$. Therefore the maximum is attained when

$$z = 3,\ 3\exp(2\pi/3),\ \text{and } 3\exp(4\pi/3).$$

For points in R we have $r < 3$, so the maximum value cannot be attained elsewhere in the disc $|z| \le 3$.

(b)(ii)

If $|z| = 3$ then using the Triangle Inequalities (Unit A1, Section 5, Para. 2)

$$|\overline{z} + 1| \le |\overline{z}| + 1 = 4 \text{ , and}$$

$$|\overline{z} - 1| \ge \big||\overline{z}| - 1\big| = |3 - 1| = 2 \text{ .}$$

Hence on the contour C we have

$$\left|\frac{\overline{z}+1}{\overline{z}-1}\exp(z^3)\right| \le \frac{4}{2} * e^{27} = 2e^{27}.$$

The length of the contour C is $2\pi * 3 = 6\pi$.

Since $\frac{\overline{z}+1}{\overline{z}-1}\exp(z^3)$ is continuous on C then, by the Estimation Theorem (Unit B1, Section 4, Para. 3), we have

$$\left|\int_C \frac{\overline{z}+1}{\overline{z}-1}\exp(z^3)\,dz\right| \le 6\pi * 2e^{27} = 12\pi e^{27}.$$

1998 Question 12

(a) 5 marks

Using the Implicit Formula (Unit D1, Section 2, Para. 11) then we have

$$\frac{(z-1)}{(z-\infty)}\frac{(2i-\infty)}{(2i-1)} = \frac{(w-i)}{(w+1)}\frac{(\infty+1)}{(\infty-i)} \Rightarrow \frac{z-1}{2i-1} = \frac{w-i}{w+1}$$

Hence $wz - w + z - 1 = 2iw + 2 - w + i$.
Rearranging gives $w(z - 2i) = -z + (3 + i)$.

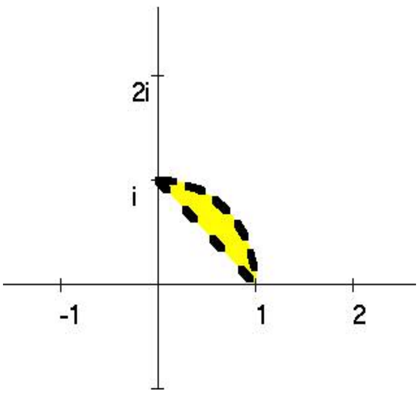
Therefore the required extended Möbius transformation is $\hat{f}$, where

$$f(z) = \frac{-z+(3+i)}{z-2i}$$

(b) 13 marks

(b)(i)

If $z = x + iy$ then
$\text{Im } z > 1 - \text{Re } z \Rightarrow y > 1 - x \Rightarrow x + y > 1.$



(b)(ii)

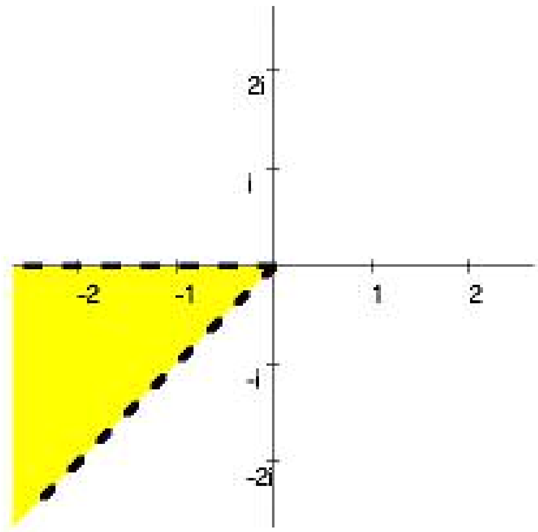
As f_1 is a Möbius transformation and R is a region in the domain of f_1 then $f_1(R)$ is a region and the boundary of R maps onto the boundary of f(R) (Unit D1, Section 4, Para. 3).

The boundary of $R = D_1 \cup D_2$, where $D_1 = \{z : \text{Re } z + \text{Im } z = 1, 0 \le \text{Re } z \le 1\}$ and $D_2 = \{z : |z| = 1, 0 \le \text{Arg } z \le \pi/2\}$

As $f_1(i) = 0$ and $f_1(1) = \infty$ then both D_1 and D_2 are mapped to extended lines which start at the origin. Since the angle between D_1 and D_2 at i is $\pi/4$ and the transformation f_1 is conformal, then the angle between the lines at the origin is also $\pi/4$.

$(1 + i)/2 \in D_1$ and $f_1\left(\frac{1+i}{2}\right) = \frac{\frac{1+i}{2} - i}{\frac{1+i}{2} - 1} = \frac{1-i}{-1+i} = -1$.

Therefore D_1 is mapped to the extended negative real axis $\{x : x \le 0\} \cup \{\infty\}$. As the interior of the region R is on the left as we travel along D_1 from i to $(i + 1)/2$, then this is also the case when we travel from f(i) to f((i + 1)/2) in the transformed region. Therefore the transformed region is as shown below.



$f(R) = \{z : -\pi < \text{Arg } z < -3\pi/4\}$

(b)(iii)

$w = g(z) = z^4$ is a one-one conformal mapping (Unit D1, Section 4, Para. 5) that maps the image of R to the upper half plane. Therefore a one-one conformal mapping from R to the upper-half plane is

$$g \circ f_1(z) = \left(\frac{z-i}{z-1}\right)^4.$$