M337/C

Third Level Course Examination 2000

Complex Analysis

Tuesday, 10 October, 2000 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I (64% of the marks) you should attempt as many questions as you can.

In Part II (36% of the marks) you should attempt no more than **TWO** questions.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.** Attach all your answer books together using the fastener provided.

The use of calculators is **NOT** permitted in this examination.

PART I

(i) You should attempt as many questions as you can in this part.

(ii) Each question in this part carries 8 marks.

Question 1

Let $\alpha = -2 - 2i$.

(a) Write down the values of

(i) $|\alpha|$,

(ii) $\operatorname{Arg}\alpha$. [2]

(b) Hence, or otherwise, evaluate each of the following, giving your answers in Cartesian form.

(i) $1/\alpha$

(ii) The principal value of $\alpha^{1/3}$

(iii) $\operatorname{Log}\alpha$

(iv) $\operatorname{Log}(\alpha^3)$ [6]

Question 2

Let $A = \{z : 1 < |z| < 2\}$, $B = \{z : 1 < |z| < 2, \quad \operatorname{Re} z > -1\}$ and $C = \{z : |z| < 1\} \cup \{z : |z| > 2\}$.

(a) Make separate sketches of A, B and C. [3]

(b) State which of the sets A, B and C are

(i) open,

(ii) a region,

(iii) a simply-connected region. [3]

(c) Using set notation, give an example of a set which is compact but not connected. [2]

Question 3

(a) Evaluate

$$\int_C \overline{z}\, dz,$$

where C is the circle $\{z : |z| = 2\}$. [3]

(b) Determine an upper estimate for the modulus of

$$\int_C \frac{\sin z}{1 + z^6}\, dz,$$

where C is the circle $\{z : |z| = 2\}$. [5]

Question 4

Let f be the function $f(z) = \dfrac{z \exp(z^2)}{(z + i)^3}$. Evaluate $\displaystyle\int_C f(z)\, dz$, where

(a) $C = \{z : |z| = \frac{1}{2}\}$; [3]

(b) $C = \{z : |z| = 2\}$. [5]

Question 5

(a) Find the residues of the function

$$f(z) = \frac{e^{2iz}}{z^2 + 9}$$

at each of the poles of f. [4]

(b) Hence evaluate the integral $\int_{-\infty}^{\infty} \frac{\cos 2t}{t^2 + 9}\,dt$. [4]

Question 6

Let $f(z) = z^5 + 5iz^3 + 7$.

(a) (i) Show that f has no zeros lying inside the circle $C_1 = \{z : |z| = 1\}$.

(ii) Determine the number of zeros of f which lie inside the circle $C_2 = \{z : |z| = 2\}$. [7]

(b) Determine an integer M such that all the zeros of f lie inside the circle $C_M = \{z : |z| = M\}$. [1]

Question 7

Let $q(z) = 3/\overline{z}$ be a velocity function on $\mathbb{C} - \{0\}$.

(a) Explain why q represents a model fluid flow. [1]

(b) Determine a complex potential function for this flow. Hence sketch the streamline through the point i and the streamline through the point $1 - i$. In each case indicate the direction of flow. [5]

(c) Evaluate the flux of q across the unit circle $\{z : |z| = 1\}$. [2]

Question 8

(a) Find the fixed points of the function $f(z) = 2z(1 - iz)$ and classify them as (super-)attracting, repelling or indifferent. [4]

(b) Exactly one of the points $-1 + \frac{1}{5}i$ and $\frac{1}{2} - i$ lies in the Mandelbrot set M.

(i) Write down which of these points is in M.

(ii) Show that the other point is not in M. [4]

PART II

(i) You should attempt at most **TWO** questions in this part.

(ii) Each question in this part carries 18 marks.

Question 9

(a) Determine for which values of the real numbers a and b the function

$$f(z) = x^2 + 2axyi + by^2, \quad \text{where } z = x + iy,$$

is analytic on $\mathbb{C}$. [7]

(b) (i) Sketch the set S in the first quadrant bounded by arcs of the circles with equations $r = 1$ and $r = 2$, and rays with equations $\theta = 0$ and $\theta = \pi/2$.

(ii) Parametrizations for two of the paths forming the boundary of S are

$$\gamma_1(t) = e^{it} \quad (t \in [0, \pi/2]) \quad \text{and} \quad \gamma_2(t) = t \quad (t \in [1, 2]).$$

Write down parametrizations γ_3 and γ_4 for the other two boundary paths.

(iii) Determine whether or not the paths γ_1, γ_2, γ_3 and γ_4 are smooth.

(iv) Show that the function Log is conformal on $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$.

(v) Determine the parametrizations of the images of the four paths from part (ii) under the function Log, and sketch the image of S under Log. [11]

Question 10

(a) Let $f(z) = \dfrac{4}{z(z-2)}$.

(i) Locate and classify the singularities of f. (No justification required.)

(ii) Determine the Laurent series about 0 for f on the set $\{z : 0 < |z| < 2\}$, giving the general term.

(iii) Determine the Laurent series about 2 for f on the set $\{z : |z-2| > 2\}$, giving the general term. [10]

(b) (i) Determine the Laurent series about 0 for the function g defined by $g(z) = z^2 \sin(1/z)$, giving the first three non-vanishing terms.

(ii) Classify the singularity of g at 0, justifying your answer.

(iii) Evaluate

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz,$$

where $C = \{z : |z| = 1\}$.

(iv) Write down the value of

$$\int_C z^{2n} \sin\left(\frac{1}{z}\right) dz, \qquad \text{for } n = 1, 2, 3, \ldots,$$

where $C = \{z : |z| = 1\}$. [8]

Question 11

(a) (i) Express e^{iz} in Cartesian form, where $z = x + iy$, and hence show that

$$|\exp(e^{iz})| = \exp(e^{-y}\cos x), \quad \text{for } z = x + iy \in \mathbb{C}.$$

(ii) Determine

$$\max\{|\exp(e^{iz})| : -\pi \leq \operatorname{Re} z \leq \pi, -1 \leq \operatorname{Im} z \leq 1\},$$

and find the point or points at which this maximum is attained. [9]

(b) Show that the functions

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n \qquad (|z| < 3)$$

and

$$g(z) = -\sum_{n=1}^{\infty}\left(\frac{3}{z}\right)^n \qquad (|z| > 3)$$

are indirect analytic continuations of each other. [9]

Question 12

(a) (i) Show that $\alpha = 1 + i$ and $\beta = 2(1 + i)$ are inverse points with respect to the circle $C = \{z : |z| = 2\}$.

(ii) Find the images of α and β under the bilinear transformation

$$g(z) = \frac{2}{z - (1 + i)},$$

and hence sketch the image of C under g.

(iii) Indicate $g(D)$ on your sketch, where $D = \{z : |z| < 2\}$. [8]

(b) Let $R = \{z : |z - 1| < 1, 0 < \operatorname{Arg}(z - 1) < \pi\}$, $R_1 = \{z_1 : \operatorname{Re} z_1 > 0, \operatorname{Im} z_1 > 0\}$ and $S = \{w : \operatorname{Im} w > 0\}$.

(i) Sketch the regions R, R_1 and S.

(ii) Find a bilinear transformation f_1 which maps R to R_1, and use the conformality of f_1 to justify that it does indeed map R to R_1.

(iii) Write down a conformal mapping from R_1 to S and hence a conformal mapping f from R to S.

(iv) Explain why the function f is not conformal on $\overline{R}$ (the closure of R). [10]

[END OF QUESTION PAPER]