M343/M

The Open University

Third Level Course Examination 1999
Applications of Probability

Monday, 18 October, 1999 2.30 pm – 5.30 pm

Time allowed: 3 hours

This examination is in **TWO** parts. Part I carries 40% of the total available marks and Part II carries 60%.

You should attempt **FOUR** questions from Part I: each question carries 10 marks. You should attempt **THREE** questions from Part II: the questions in this part carry 20 marks each.

Write your answers to Parts I and II in **separate** answer books. Please start each question on a new page, and cross out rough working.

At the end of the examination

Fasten together your answer books for Parts I and II, using the paper fastener provided. Check that you have written your personal identifier and examination number on **each** answer book used. **Failure to do so will mean that your work cannot be identified.**

Important note

Note that if you have to solve an equation (or set of equations) in any question, then you should show all the intermediate steps in your solution, whether you solve the equation(s) algebraically or using an iterative procedure. If you simply state an answer without including such details, then *you will not be given any credit for the answer.*

PART I (Questions 1 to 6)

You should attempt **FOUR** *questions from this part of the examination, which carries 40% of the total available marks. Each question carries 10 marks.*

Question 1

The positions of a certain species of wild flower in a wood may be assumed to be reasonably modelled by a two-dimensional Poisson process with density $\lambda = 0.15$ per square metre.

(i) A flower is selected at random.

(a) What is the expected distance to the flower nearest to it?

(b) Find the probability that the nearest flower is more than 2 metres away.

(ii) A rectangular region 4 metres long by 3 metres wide is cordoned off.

(a) Find the probability that there are no flowers in this region.

(b) Find the probability that there are more than two flowers in this region.

(iii) Simulate the number of flowers in a region 4 metres long by 3 metres wide, using the number $u = 0.8872$ which is an observation from the uniform distribution $U(0, 1)$.

Question 2

The offspring random variable X in a Galton-Watson branching process starting with a single individual in generation zero has the probability generating function

$$\Pi(s) = \frac{2 + 3s}{8 - 3s}.$$

(i) Identify the distribution of X and hence, or otherwise, find the mean number of individuals in the first generation.

(ii) Find to four decimal places the probability that the process becomes extinct at the third generation.

(iii) Calculate the probability that the process eventually becomes extinct. How could you have known without calculating its value that this probability is less than one?

Question 3

A partial differential equation for the probability generating function $\Pi(s, t)$ of the integer-valued random variable $X(t)$, which denotes the number of individuals alive at time t in a simple birth process $\{X(t); t \geq 0\}$ with birth rate β, is

$$\frac{\partial \Pi}{\partial t} = -\beta s(1 - s)\frac{\partial \Pi}{\partial s}.$$

(i) Show that the general solution of this partial differential equation may be written as

$$\Pi(s, t) = \psi\left(\frac{s}{1 - s}e^{-\beta t}\right),$$

where ψ is an arbitrary function.

(ii) Suppose that there are two individuals alive at time 0, when observation of the process commences. Find the particular solution corresponding to this condition.

Question 4

One of a family of six comes home with a cold; the other five members of the family are all susceptible to the cold. Thereafter the spread of the cold through the family can be described by a stochastic epidemic model. The infective contact rate is $\beta = 1.25$ per day and any person who catches the cold is infectious for a time which is exponentially distributed with mean 4 days.

(i) Show that the epidemic parameter ρ is equal to 1.

(ii) Find the probability that two or fewer of the five initially uninfected members of the family catch the cold.

Question 5

The lifetime distribution of members of a certain stationary insect population has probability density function

$$f(x) = \frac{1}{25} x e^{-x^2/50}, \quad x \geq 0,$$

where x is measured in days.

(i) Identify the lifetime distribution of an individual and hence write down the expectation of life at birth.

(ii) Find the life table function $Q(x)$ for members of this population.

(iii) Find the median lifetime for insects in the population.

(iv) Find the mean age of insects in this population.

(v) What proportion of insects live for more than 10 days?

Question 6

The value of a painting fluctuates in such a way that the random variable

$$V(t) = \frac{\text{value at time } t}{\text{value at time } 0}$$

may be modelled as a geometric Brownian motion, derived from ordinary Brownian motion $\{X(t); t \geq 0\}$ with diffusion coefficient $\sigma^2 = \frac{1}{8}$ per year through the relationship $V(t) = \exp X(t)$.

(i) Find the probability that after six months the value of the painting will be more than 25% up on its value at time 0.

(ii) Suppose that after a year the value of the painting is the same as the value at time 0. Find the probability that after four months the value of the painting was less than 75% of its value at time 0.

PART II (Questions 7 to 12)

You should attempt **THREE** *questions from this part of the examination, which carries 60% of the total available marks. Each question carries 20 marks: the mark allocation for each part of each question is shown beside each part thus:* [*4*].

Question 7

(a) (i) Give one example of a random phenomenon which could reasonably be modelled as a compound Poisson process. Explain briefly why a compound Poisson process is an appropriate model. [*2*]

(ii) Give one example of a random phenomenon which could reasonably be modelled as a multivariate Poisson process. Explain briefly why a multivariate Poisson process is an appropriate model. [*2*]

(b) A computer programmer learning a new application notices that she makes serious errors less frequently as she becomes more experienced with the application. A reasonable model for the incidence of serious errors after t hours of experience with the application is provided by a non-homogeneous Poisson process with rate

$$\lambda(t) = \frac{3}{2+3t}, \quad t \geq 0.$$

(i) Show that the expected number of serious errors made in the first t hours using the application is $\mu(t) = \log(1 + \frac{3}{2}t)$. [*2*]

(ii) Find the probability that the programmer does not make any serious errors in her first hour using the application. [*2*]

(iii) Find the probability that the programmer makes at least one serious error in her second hour using the application. [*3*]

(iv) Find the probability that the first time that the programmer makes a serious error is during her second hour using the application. [*3*]

(v) Show that the simulated times $t_1, t_2, \ldots$, at which the programmer makes serious errors may be obtained from the recurrence relation

$$t_{j+1} = \frac{3t_j + 2u}{3(1-u)},$$

where u is a random observation from the uniform distribution $U(0,1)$. [*3*]

(vi) Simulate the times at which the programmer makes her first two serious errors. Use the numbers $u_1 = 0.23735$, $u_2 = 0.49200$, which are random observations from the uniform distribution $U(0,1)$. Give the times to the nearest minute. [*3*]

Question 8

(a) Describe one example of a situation which could reasonably be modelled by a particle executing a simple random walk on the line with two reflecting barriers. Explain briefly why a simple random walk might be a suitable model and what the reflecting barriers represent. [3]

(b) A particle executes a simple unrestricted random walk on the line, a step to the right of length 1 occurring with probability p, and a step to the left of length 1 occurring with probability $q = 1 - p$. Its position at time n is represented by the random variable $X_n, n = 0, 1, 2, \ldots$. Its initial position is given by $X_0 = 0$.

(i) Show that the distribution of X_n is given by

$$P(X_n = k) = \binom{n}{(n+k)/2} p^{(n+k)/2} q^{(n-k)/2}$$

for $k \in \{-n, -n+2, -n+4, \ldots, n-4, n-2, n\}$. [4]

(ii) Find in terms of p and q the values of the probabilities

$$P(X_7 = -2), \quad P(X_7 = 1).$$ [2]

(iii) Find in terms of p and q the probability that the particle returns to the origin for the first time after 6 steps. [2]

(iv) When $p = \frac{2}{3}$, find an approximate value for the probability that the particle is more than 100 steps to the right of its starting point after 450 steps. [6]

(v) When $p = \frac{2}{3}$, find the probability that, starting from the origin, the particle visits the point -3 before it visits the point $+2$. [3]

Question 9

(a) (i) Explain briefly why an unrestricted random walk on the line is a Markov chain. [2]

(ii) Give an example of a Markov chain which is not a random walk. Explain briefly why your example is not a random walk. [2]

(b) In a clinical study of mood, the following Markov chain model was one of several tested. A patient's mood was assessed at noon each day as either happy (H), apathetic (A) or depressed (D). After analysis of records, the following transition matrix was suggested as a model for the daily mood changes.

$$\mathbf{P} = \begin{matrix} H \\ A \\ D \end{matrix} \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(i) If a patient is depressed on Monday, according to the model what is the probability that he will be depressed on Wednesday? If a patient is depressed on Monday, find the probability that he will be apathetic on Tuesday and happy on Wednesday. [4]

(ii) According to the model, what proportion of days in the long run is a patient in each of the three moods? [6]

(iii) If a patient is happy today, what is the expected number of days until he is next happy? [1]

(iv) On admission, $\frac{3}{4}$ of patients are assessed as depressed and the rest as apathetic. What proportion of patients are depressed one day after admission? What proportion are depressed two days after admission? What proportion of patients are depressed twenty days after admission? [5]

Question 10

Two advisers are on duty at a tax enquiry office. Members of the public seeking advice may be assumed to arrive independently and at random at an average rate of 16 per hour. If an adviser is free, then an arriving enquirer is seen immediately; otherwise a central queue is formed. The time spent by either adviser with an enquirer may be assumed to be exponentially distributed with mean 6 minutes.

(i) Write down the specification for this queue, and show that the traffic intensity ρ is 0.8. *[2]*

Assume that the queue is in equilibrium.

(ii) Find the proportion of the time that both advisers are idle. *[3]*

(iii) What proportion of enquirers are seen immediately? *[4]*

(iv) Find the proportion of the time that there are more than three people waiting to be seen. *[5]*

(v) Show that the probability generating function of the equilibrium queue size is given by

$$\Pi(s) = \frac{1}{45}\left(5 + 8s + \frac{32s^2}{5 - 4s}\right).$$

Hence evaluate the mean equilibrium queue size. *[6]*

Question 11

(a) A certain blood condition may be thought of as being due to a single recessive allele **a**. The 'normal' allele **A** is dominant. A population in which this blood condition occurs is in Hardy-Weinberg equilibrium, the alleles **A** and **a** occurring in the proportions 0.9:0.1. Mating is at random.

(i) What proportion of individuals

(a) are normal (genotype **AA**),

(b) are carriers (genotype **Aa**),

(c) have the blood condition? *[3]*

(ii) Given that an individual does not have the blood condition, find the probability that he or she is

(a) a carrier (**Aa**),

(b) normal (**AA**). *[4]*

Ralph and Susan, neither of whom has the blood condition, have two children Tom and Ursula.

(iii) Find the probability that Tom has the blood condition. *[4]*

(iv) Find the probability that Tom has the blood condition *and* Ursula does not. *[3]*

(b) Now suppose that the blood condition is due to a single recessive allele **b** which is carried on the X chromosome; the normal allele **B** is dominant. The allele proportions may be assumed to be the same in each sex, the alleles **B** and **b** occurring in the proportions 0.9:0.1. Mating is at random.

Harry and Irene, neither of whom has the blood condition, have a son James.

(i) What is the probability that James has the blood condition? *[3]*

(ii) Given the additional information that James does not have the blood condition, find the probability that his mother Irene is a carrier (**Bb**). *[3]*

Question 12

(a) Describe in non-technical language what is meant by the term 'new better than used' (NBU) applied to a component in a mechanical system. [2]

(b) The lifetime T (in years) of a thermocouple in a domestic boiler may be described in terms of its hazard function

$$h(t) = \frac{2t}{25 - t^2}, \quad 0 \leq t < 5.$$

(i) Find the survivor function for the thermocouples. What proportion of thermocouples last for less than two years? [4]

(ii) Find the mean lifetime of a thermocouple. [2]

Whenever a thermocouple fails, it is immediately replaced. Suppose that, at some time when the boiler is quite old, new occupiers move into a house where one of these thermocouples is in use.

(iii) Find the probability that this thermocouple fails within a year. [4]

(iv) Find the p.d.f. of the total lifetime of this thermocouple, and calculate its expected total lifetime. Explain why the mean total lifetime of the thermocouple in use when the new occupiers move in is longer than the mean lifetime of a new thermocouple chosen at random and fitted in the boiler. [6]

(v) The new occupiers decide not to replace the boiler for at least six years. What is the expected number of new thermocouples that will be fitted during their first six years in the house? [2]

[END OF QUESTION PAPER]