M343/H

The Open University

Third Level Course Examination 1998

Applications of Probability

Thursday, 15 October, 1998 2.30pm – 5.30pm

Time allowed: 3 hours

This examination is in **TWO** parts. Part I carries 40% of the total available marks and Part II carries 60%.

You should attempt **FOUR** questions from Part I: each question carries 10 marks. You should attempt **THREE** questions from Part II: the questions in this part carry 20 marks each.

Write your answers to Parts I and II in **separate** answer books. Please start each question on a new page, and cross out rough working.

At the end of the examination

Fasten together your answer books for Parts I and II, using the paper fastener provided. Check that you have written your personal identifier and examination number on **each** answer book used. **Failure to do so will mean that your work cannot be identified.**

PART I (Questions 1 to 6)

You should attempt **FOUR** *questions from this part of the examination, which carries 40% of the total available marks. Each question carries 10 marks.*

Question 1

Customers arrive at a small art gallery during its opening hours according to a Poisson process with mean interarrival time 20 minutes. Some customers purchase one or more pictures, others do not purchase any: Y, the number of purchases per customer, has a geometric distribution, $G_0\left(\frac{1}{10}\right)$.

(i) Find the mean and variance of the number of pictures purchased per customer.

(ii) Find the mean and variance of the total number of pictures purchased in six hours.

(iii) Find the index of dispersion for this process. What does the index of dispersion tell you about the occurrences of picture purchases?

Question 2

A particle executes an unrestricted simple random walk on the line with $p = 0.7$ and $q = 0.3$. Its position at time n is represented by the random variable X_n, $n = 0, 1, 2, \ldots$. Its initial position is given by $X_0 = 0$.

(i) Find the values of the following probabilities.

(a) $P(X_5 = -2)$ (b) $P(X_6 = 2)$

(ii) Find the probability that the particle returns to its starting point for the first time after 8 steps.

(iii) After the particle has been moving for some time, it is observed to be located at the point $+3$. Find the probability that it ever returns to the origin.

Question 3

A six-state Markov chain has the following transition matrix.

$$\mathbf{P} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) If the Markov chain is in state 3 at time 0, find the probability that it is again in state 3 at time 2.

(ii) If the Markov chain is in state 3 at time 0, find the probability that it is in state 5 at time 1 and in state 4 at time 2.

(iii) Classify the states of the Markov chain, identifying communicating classes, and stating whether each class is closed or not closed, recurrent or transient, and periodic or aperiodic; if periodic, state its period.

Question 4

(a) Characterize the following queues according to the standard notation.

(i) Vehicles arrive at random at a filling station where there are four pumps. The time taken for each driver to serve himself and pay is equally likely to be anything from four to seven minutes.

(ii) Patients arrive at a dentist's surgery every ten minutes; there is one dentist on duty. The time spent with each patient has a gamma distribution with mean seven minutes.

(b) Customers arrive at random at an average rate of two every five minutes at a small post office where there is one cashier. Individual service times are exponentially distributed with mean two minutes.

(i) Calculate the traffic intensity of this queue.

The pattern of arrivals at and departures from the post office may for practical purposes be assumed to attain equilibrium shortly after the start of business each day.

(ii) What proportion of customers arrive at the post office to find precisely one customer ahead of them, the one currently being served?

(iii) At any time, what is the mean number of customers in the post office, either being served or waiting to be served?

(iv) What is the probability distribution of the total time in minutes from a customer's arrival until her departure?

Question 5

The waiting time T between successive occurrences of an event E in a discrete-time renewal process has the probability distribution

$$P(T = 1) = 0.7, \quad P(T = 2) = 0.3.$$

(i) Find the generating function $U(s)$ for this process, and hence or otherwise find the probabilities u_2 and u_3.

(ii) The waiting time to the sixth occurrence of E is denoted by W_6. Find the probability $P(W_6 = 10)$.

(iii) If an observer arrives after the renewal process has been running for a long time, what is the approximate probability that an event occurs at the next time point?

Question 6

The variation in the price of a particular share may be modelled as a Brownian motion with drift. The drift coefficient is 10 pence per month and the diffusion coefficient is 400 (pence)^2 per month. Originally the cost of the share to its purchaser was £2.50. So $C(t)$, the price of the share t months after its purchase, is given by

$$C(t) = 250 + 10t + X(t),$$

where $X(t)$ is ordinary Brownian motion with diffusion coefficient 400 (pence)^2 per month.

(i) Find the probability that one month after purchase the price of the share is lower than when it was bought.

(ii) It is known that four months after it was bought the share was trading at £3.00. Find the probability that three months earlier (that is, one month after its purchase) its price was less than £2.50.

PART II (**Questions 7 to 12**)

You should attempt **THREE** *questions from this part of the examination, which carries 60% of the total available marks. Each question carries 20 marks: the mark allocation for each part of each question is shown beside each part thus:* [*4*].

Question 7

(a) Give an example of objects distributed in two-dimensional space for which a two-dimensional Poisson process would *not* provide a good probability model, and say why the model would not be appropriate. Suggest a model *other than the two-dimensional non-homogeneous Poisson process* which you consider to be more suitable, giving a brief explanation for your choice of model. [*4*]

(b) The positions of a particular species of small shrub in a certain area of countryside may be assumed to be reasonably modelled by a two-dimensional Poisson process with density $\lambda = 120$ per square kilometre.

(i) What is the expected distance from a randomly selected shrub to the one nearest to it? Find the probability that the nearest shrub is less than 30 metres away. [*4*]

A large part of the area is divided into regions 200 metres long and 150 metres wide. A region is selected at random.

(ii) Find the probability that the region contains at least two of the shrubs. [*3*]

(iii) Simulate the number of this species of shrub in the region using the number $u = 0.3907$, which is a random observation from the uniform distribution, $U(0,1)$. [*3*]

(iv) *Without carrying out the simulation*, explain clearly how you would simulate the positions of these shrubs in the region. [*3*]

(v) In any year, 30% of shrubs of this species produce flowers. Explain how you would continue the simulation to determine the positions in the region of the shrubs that flower next year. [*3*]

Question 8

(a) The offspring random variable X in a population developing as a Galton-Watson branching process has probability distribution $P(X = x) = p_x$, $x = 0, 1, 2, \ldots$, and mean $\mu_X = \mu$. In each of the following cases, briefly describe what might eventually happen to the evolving population.

(i) $p_0 = 0$ (ii) $p_0 > 0$, $\mu < 1$

(iii) $p_0 > 0$, $\mu = 1$ (iv) $p_0 > 0$, $\mu > 1$ [*4*]

(b) In a Galton-Watson branching process starting with a single individual in generation zero, the offspring distribution is binomial, $B\left(3, \frac{2}{3}\right)$.

(i) Write down the mean and variance of the number of individuals in the first generation, and calculate the mean and variance of the number of individuals in the second generation. [*4*]

(ii) Find the probability that the process becomes extinct at the third generation; give your answer to 4 decimal places. [*5*]

(iii) Calculate to 4 decimal places the probability that the process eventually becomes extinct. [*5*]

(iv) If there were three individuals in generation zero, what would be the probability that the process eventually becomes extinct? [*2*]

Question 9

A partial differential equation for the probability generating function $\Pi(s,t)$ of the integer-valued random variable $X(t)$ denoting the number of individuals alive at time t in a pure death process $\{X(t); t \geq 0\}$ is

$$\frac{\partial \Pi}{\partial t} = \nu(1-s)\frac{\partial \Pi}{\partial s},$$

where ν is the individual death rate.

(i) Solve this equation using Lagrange's method to obtain the probability generating function $\Pi(s,t)$ of the random variable $X(t)$, given that there are six individuals alive at time 0. Hence identify the probability distribution of $X(t)$. [12]

(ii) What is the expected time until the first death occurs? Find the expected time until the population dies out. [4]

(iii) What is the probability that the population will not yet have died out at time t? [4]

Question 10

(a) Briefly describe the main similarities and differences between the threshold phenomena for the stochastic general epidemic model and the deterministic general epidemic model. [4]

(b) In a household of students of total size 7, three students return from a holiday suffering from a disease which is incurable but not serious; the other four students are susceptible. Infectives are not isolated and the infective contact rate is $\beta = 0.9$ per day.

(i) Suggest a stochastic model which may be used to represent the spread of the disease in the household. [1]

(ii) Find the mean and standard deviation of the waiting time until all seven students have the disease. [6]

(c) In a hostel containing 31 students, three students return from a field trip with a different disease; the other 28 students are susceptible. An individual who catches this disease remains infectious for a time which is exponentially distributed with mean 4 days; they cannot catch the disease a second time. The infective contact rate is $\beta = 1.5$ per day.

(i) Show that the epidemic parameter ρ is equal to 5. [1]

(ii) Use a stochastic model to calculate the probability that one and only one of the susceptible students catches the disease before the epidemic dies out. [6]

(iii) Find an approximate value for the probability that a major outbreak of the disease occurs. [2]

Question 11

(a) Give one example each of a population for which a stationary population model (i) might be suitable, and (ii) would not be suitable. For example (i), say what assumptions you are making about your population by proposing a stationary population model. For example (ii), explain briefly why a stationary population model would not be suitable. [6]

(b) The age-specific death rate for the members of a particular stationary population is given by the formula

$$h(x) = \frac{3}{2+x}, \quad x \geq 0,$$

where x is measured in years.

(i) Find the life table function $Q(x)$ for members of this population. [4]

(ii) Calculate the mean lifetime for members of this population. [2]

(iii) Write down the p.d.f. $g(x)$ for the age-distribution of members of this population, and hence find the proportion $G(x)$ of the population that are aged x or younger. [3]

(iv) (a) What proportion of members of the population are over 3 years old?

(b) What proportion of individuals live for longer than 3 years? [2]

(v) Find the mean age of members of this population. [3]

Question 12

The eye colour of a particular species of animal is controlled by a single gene with two alleles **A** and **a**. Homozygotes **AA** have yellow eyes, heterozygotes **Aa** have green eyes and homozygotes **aa** have blue eyes. In a large population of these animals, reproducing in discrete generations by a process of random mating, 36% of the animals have blue eyes.

(i) What proportion of the animals have yellow eyes? What proportion of the animals have green eyes? [4]

All the yellow-eyed animals are removed from a large population of these animals in an attempt to breed only blue-eyed and green-eyed animals. The remaining animals form generation zero of a new population.

(ii) Assuming that mating habits remain unaltered, explain what happens to the proportions of yellow-eyed, green-eyed and blue-eyed animals in future generations. Mention any theorem that you use. [9]

(iii) Two animals selected at random from the first generation of animals are mated. Their first offspring is green-eyed. What is the probability that both parent animals are green-eyed? [7]

[END OF QUESTION PAPER]