M343/A

The Open University

Third Level Course Examination 1996

Applications of Probability

Monday, 14 October, 1996 10.00 am – 1.00 pm

Time allowed: 3 hours

This examination is in **TWO** parts. Part I carries 40% of the total available marks and Part II carries 60%.

You should attempt **FOUR** questions from Part I: each question carries 10 marks. You should attempt **THREE** questions from Part II: the questions in this part carry 20 marks each.

Write your answers to Parts I and II in **separate** answer books. Please start each question on a new page, and cross out rough working.

At the end of the examination

Fasten together your answer books for Parts I and II, using the paper fastener provided. Check that you have written your name, personal identifier and examination number on **each** answer book used. **Failure to do so will mean that your work cannot be identified.**

PART I **(Questions 1 to 6)**

You should attempt **FOUR** *questions from this part of the examination, which carries 40% of the total available marks. Each question carries 10 marks.*

Question 1

Telephone calls are received at an advice bureau desk according to a Poisson process at a rate of one call every fifteen minutes. The number of questions asked during each call has a geometric distribution (starting at 1) with mean 2.5.

(i) Obtain the probability generating function of the number of questions asked per telephone call. What proportion of callers ask more than two questions?

(ii) Find the probability generating function of the total number of questions put to an adviser in a half-hour period.

(iii) Find the mean number of questions asked in a half-hour period.

Question 2

Each weekend, Eliza relaxes by either reading a book (B), doing a jigsaw (J) or sewing (S). What she chooses to do on any weekend depends on what she did the previous weekend: the pattern of weekend activities may be modelled by a Markov chain with the following transition matrix.

$$\mathbf{P} = \begin{matrix} B \\ J \\ S \end{matrix} \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

(i) If she does a jigsaw one weekend, find the probability that she sews the next weekend and reads the weekend after that.

(ii) According to this model, what proportion of weekends in the long run does Eliza spend on each of the three activities?

(iii) If she reads a book this weekend, what is the expected number of weeks until she next reads a book?

Question 3

Customers arrive at random at an average rate of 30 per hour at a post office where there are three cashiers on duty. When all the cashiers are busy, a central queue is formed. Individual service times are exponentially distributed with mean 4 minutes.

(i) Write down the specification of this queue and calculate the traffic intensity ρ.

(ii) Assuming that the queue is in equilibrium, find

(a) the proportion of the time that all three cashiers are idle;

(b) the mean number of cashiers busy at any particular time.

Question 4

The spread of a cold through a hostel of 31 people is to be modelled as a deterministic general epidemic. Anyone who catches the cold eventually recovers and is thereafter immune. Initially there are 6 infectives; the other 25 residents are all susceptible to the cold. The infectious contact rate is 0.75 per day and the recovery rate is 0.4 per day.

(i) Show that the epidemic parameter ρ is equal to 16.

(ii) Find the maximum number of residents who are ill at any one time.

(iii) When the epidemic is over, how many residents have had the cold and recovered?

(iv) Sketch the trajectory of the epidemic, showing the starting coordinates, the coordinates at which the epidemic is at its height and the end coordinates.

Question 5

The waiting time T between successive occurrences of an event E in a discrete-time renewal process has the probability distribution

$$P(T = 2) = 0.5, \qquad P(T = 3) = 0.5.$$

(i) Find the generating function $U(s)$ for this process and hence or otherwise find the probabilities u_4, u_5 and u_6.

(ii) The waiting time to the fifth renewal is denoted by W_5.

(a) Find the range of W_5.

(b) Find the probability $P(W_5 = 13)$.

Question 6

The height above sea level of a migrating bird (of a certain species) when crossing the English Channel fluctuates in such a way that the random variable

$$H(t) = \frac{\text{height at time } t}{\text{height at time } 0}$$

may be modelled as a geometric Brownian motion, derived from ordinary Brownian motion $\{X(t); t \geq 0\}$ with diffusion coefficient $\sigma^2 = 0.08$ per hour through the relationship $H(t) = \exp X(t)$. A bird crosses the English coast at time 0 at a height of 25 metres above sea level.

(i) Find the probability that after half an hour the bird is more than 30 metres above sea level.

(ii) When the bird crosses the French coast three hours after leaving the English coast, it is observed to be 25 metres above sea level. Find the probability that when it passed a ship two hours after leaving the English coast, it was less than 20 metres above the sea.

PART II (Questions 7 to 12)

You should attempt **THREE** *questions from this part of the examination, which carries 60% of the total available marks. Each question carries 20 marks: the mark allocation for each part of each question is shown beside each part thus:* [4].

Question 7

(a) Suggest one example each of objects located in two-dimensional space whose disposition could not sensibly be modelled as a two-dimensional Poisson process (i) because they tend to group in clusters, (ii) because they tend to be rather too regularly located in relationship to one another. For each example, suggest a model *other than the two-dimensional non-homogeneous Poisson process* which you consider to be more suitable, giving a brief explanation for your choice of model. [6]

(b) In an area consisting mainly of grassland, bushes may be assumed to be randomly located according to a two-dimensional Poisson process with density $\lambda = 4 \times 10^{-5}$ per square metre.

(i) What is the expected distance from a randomly selected bush to the bush nearest to it? Find the probability that the nearest bush is less than 100 metres away. [4]

A botanical survey of the area is carried out. The area is divided into regions 400 metres long and 250 metres wide.

(ii) Find the probability that a randomly selected region contains at least one bush. What is the probability that a region contains no more than two bushes? [4]

(iii) Simulate the number of bushes in a region using the number $u = 0.3920$, which is a random observation from the uniform distribution, $U(0, 1)$. [3]

(iv) *Without carrying out the simulation*, explain how you would simulate the positions of these bushes in the region. [3]

Question 8

(a) Describe one example of a situation which could reasonably be modelled by a particle executing a simple random walk on the line with two absorbing barriers. [3]

(b) A particle executes an unrestricted random walk on the line starting at the origin; its position after n steps is denoted X_n. The ith step, Z_i, has the following distribution:

$$P(Z_i = 4) = p, \qquad P(Z_i = -1) = q = 1 - p.$$

(i) Find the mean and variance of Z_i, and hence write down the mean and variance of X_n. [4]

(ii) Derive the distribution of X_n; that is, find the probability $P(X_n = x)$, and state the range of X_n. Explain all the steps in your derivation. [5]

(iii) Find the values of the following probabilities:

(a) $P(X_7 = -3)$, (b) $P(X_6 = 4)$. [2]

(iv) Find the probability that the particle returns to the origin for the first time after 5 steps. [2]

(v) When $p = \frac{1}{2}$, find the value of the probability $P(X_{10} > 0)$. [4]

Question 9

Swimmers arrive at a large swimming pool according to a Poisson process at a rate of λ per hour. Each swimmer, independently of the others, stays for a time which is exponentially distributed with mean $1/\nu$ hours. A partial differential equation for the probability generating function $\Pi(s,t)$ of $X(t)$, the number of swimmers in the pool t hours after it opens for the day, is

$$\frac{\partial \Pi}{\partial t} = -\lambda(1-s)\Pi + \nu(1-s)\frac{\partial \Pi}{\partial s}.$$

(i) Rewrite this equation in Lagrange form and identify the functions f, g and h. [1]

(ii) Write down the auxiliary equations, and show that their two solutions may be written

$$c_1 = (1-s)e^{-\nu t}, \qquad c_2 = \Pi e^{-\lambda s/\nu}.$$ [7]

(iii) Write down the general solution for the partial differential equation for $\Pi(s,t)$. [1]

(iv) Show that the particular solution for $\Pi(s,t)$, given that initially the swimming pool is empty, is

$$\Pi(s,t) = \exp\left(-\frac{\lambda}{\nu}(1-e^{-\nu t})(1-s)\right)$$

and hence identify the distribution of $X(t)$. [5]

(v) If swimmers arrive at an average rate of six per minute and stay for twenty minutes on average, find the mean number of swimmers in the pool when it has been open for several hours. [2]

(vi) The pool is closed to new swimmers at five in the afternoon. At six o'clock any remaining swimmers are asked to leave. If there are 50 swimmers in the pool at five o'clock, find the probability that there are still at least two swimmers in the pool at six o'clock. [4]

Question 10

(a) Briefly explain what is meant by the terms *stationary population* and *stable population*. Describe how a population which is stable but not stationary might evolve. [5]

(b) The age-specific death rate for the members of a particular stationary population is given by

$$h(x) = \frac{1}{90-x}, \qquad 0 \le x < 90.$$

(i) Obtain the life table function $Q(x)$ for members of the population. [3]

(ii) Find the mean lifetime for members of the population. [2]

(iii) Find the mean age of members of the population. [3]

(iv) Draw a sketch of $g(x)$, the p.d.f. of the age-distribution. [1]

(v) Suppose that, in a population with the life table function of part (i), there are equal numbers of males and females and that the age-distribution is the same for males and females.

(a) Draw a rough diagram showing what a population pyramid would look like if the population is stationary.

(b) Now suppose that the population is stable but not stationary. Describe briefly how the shape of the population pyramid would differ from that for the stationary population in part (v)(a) if the population is (i) growing, (ii) declining. Draw rough diagrams to illustrate your answer. [6]

Question 11

In a large isolated population of animals, eye colour is controlled by a single gene. Doubly dominant animals **BB** and heterozygotes **Bb** have blue eyes; homozygous recessives **bb** have green eyes. The population has been breeding for many generations. Mating may be assumed to be at random. It is observed that only 36% of animals have blue eyes.

(i) What proportion of the animals is 'pure' blue-eyed (doubly dominant **BB**)? [4]

(ii) What proportion of the blue-eyed animals is 'pure' blue-eyed? [2]

The blue-eyed animals are separated from the green-eyed animals. The blue-eyed animals form generation 0 of a new population. They are allowed to mate at random.

(iii) Explain what will happen to the proportions of blue- and green-eyed animals in future generations of this population, mentioning any theorem that you use. [8]

(iv) Two animals selected at random from the first generation of animals are mated. Their first offspring is blue-eyed. What is the probability that both parent animals are 'pure' blue-eyed? [6]

Question 12 X

(a) Sketch the autocorrelation function for a moving average process of order 1. Describe a random phenomenon which might reasonably be modelled by a MA(1) process. [5]

(b) The AR(2) process $\{X_t; t = 0, 1, 2, \ldots\}$ is defined by

$$X_t = 0.5X_{t-1} + \alpha X_{t-2} + Z_t,$$

where $\alpha \neq 0$, $\{Z_t\}$ is a purely random process with mean 0 and variance 9, and each Z_t is independent of the preceding values of X.

(i) Between what bounds must α lie for the process $\{X_t\}$ to be weakly stationary? [2]

Suppose that all conditions for weak stationarity are satisfied and that $\alpha = 0.36$.

(ii) Find the values of ρ_0 and ρ_1. [2]

(iii) Find the autocorrelation function of the process. [7]

(iv) The first two points X_0, X_1 in a realization of the process are 2.17, −0.62 respectively. Simulate the next point X_2 using the number 0.933, which is a random observation from a standard normal distribution. [2]

(c) Describe briefly the way in which moving average processes and autoregressive processes are related to one another. [2]

[END OF QUESTION PAPER]